Kinetic theory and transport equation

Kinetic theory

- Find the distribution function of energetic particles

$$f(\mathbf{r},\mathbf{p},t)\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{p}$$

- Liouville equation for the total phase space density, $\mathcal{D}(\mathbf{r}_1,\mathbf{p}_1,\ldots\mathbf{r}_N,\mathbf{p}_N)\mathrm{d}\mathbf{r}_1\ldots\mathrm{d}\mathbf{r}_N\mathrm{d}\mathbf{p}_1\ldots\mathrm{d}\mathbf{p}_N$

$$\frac{\partial\mathcal{D}}{\partial t}+\nabla_{6N}\cdot(D\mathcal{V}_{6N})=0$$

phase space density conservation

r and p are conjugate variables

$$\frac{\partial\mathcal{D}}{\partial t}+\sum_{i=1}^{N}\left[\dot{\mathbf{r}}_i\cdot\frac{\partial\mathcal{D}}{\partial\mathbf{r}_i}+\dot{\mathbf{p}}_i\cdot\frac{\partial\mathcal{D}}{\partial\mathbf{p}_i}\right]=0$$

Kinetic theory

- Distribution functions $f(\mathbf{r},\mathbf{p},t)\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{p}$

1 particle $f_1(\mathbf{r},\mathbf{p},t) = N\int \mathcal{D}\,\mathrm{d}\mathbf{r}_2\mathrm{d}\mathbf{p}_2\ldots\mathrm{d}\mathbf{r}_N\mathrm{d}\mathbf{p}_N$

2 particles $f_{12}(\mathbf{r}_1,\mathbf{p}_1;\mathbf{r}_2,\mathbf{p}_2;t) = N(N-1)\int \mathcal{D}\,\mathrm{d}\mathbf{r}_3\mathrm{d}\mathbf{p}_3\ldots\mathrm{d}\mathbf{r}_N\mathrm{d}\mathbf{p}_N$

- BBGKY hierarchy…

$$-\int \mathbf{X}_{12}\cdot(\nabla_{\mathbf{p}1}f_{12})d\mathbf{r}_2 d\mathbf{p}_2$$

$$\frac{\partial f_1}{\partial t} + \dot{\mathbf{r}}_1\cdot\frac{\partial f_1}{\partial \mathbf{r}_1} + \mathbf{X}_1\cdot\frac{\partial f_1}{\partial \mathbf{p}_1} = C(f_{12})$$

force felt by particle 1

- Vlasov equation
  (collisionless: long-distance interactions)

$$\frac{\partial f_1}{\partial t} + \dot{\mathbf{r}}_1\cdot\frac{\partial f_1}{\partial \mathbf{r}_1} + (\mathbf{X}_1 + \mathbf{X}_1')\cdot\frac{\partial f_1}{\partial \mathbf{p}_1} = 0$$

includes contribution of all particles (mean field)

The use of kinetic theory

- Separation of energetic particles and plasma:
  - energetic particles considered as a distinct class of particles, with their own distribution function, interacting with an “external” medium: the interstellar plasma
  - The underlying plasma holds stochastic, inhomogeneous magnetic fields (turbulence, waves…)

$$f(\mathbf{r},\mathbf{p},t)\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{p}$$

NB: the influence of energetic particles on the plasma must be negligible

$$\frac{\partial f}{\partial t}+\dot{\mathbf{r}}\cdot\frac{\partial f}{\partial \mathbf{r}}+q(\mathbf{E}+\mathbf{p}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{p}}=0$$

$$q(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{p}} = q\left[(\mathbf{v}-\mathbf{u})\times\mathbf{B}\right]\cdot\frac{\partial f}{\partial \mathbf{p}}$$

$$= -\mathbf{B}\cdot\left[q(\mathbf{v}-\mathbf{u})\times\frac{\partial}{\partial \mathbf{p}}\right] f \equiv -\mathbf{B}\cdot\hat{\mathbf{D}} f$$

$$\boxed{\frac{\partial f}{\partial t} + \dot{\mathbf{r}}\cdot\frac{\partial f}{\partial \mathbf{r}} - \mathbf{B}\cdot\hat{\mathbf{D}} f = 0}$$

- Average over magnetic fields

$$\frac{\partial \bar{f}}{\partial t} + \mathbf{v}\cdot\frac{\partial \bar{f}}{\partial \mathbf{r}} - \mathbf{B}_0\cdot\hat{\mathbf{D}}\bar{f} = \hat{D}_\alpha \bar{T}_{\alpha\beta}\hat{D}_\beta \bar{f}$$

where $\bar{T}_{\alpha\beta}(\mathbf{r},\mathbf{v}) = \int_0^\infty \mathrm{d}\tau T_{\alpha\beta}(\mathbf{r};(\mathbf{v}-\mathbf{u})\tau)$

and $T_{\alpha\beta}(\mathbf{r},\mathbf{r}')$ is the correlation tensor of the stochastic magnetic field

The diffusive approximation

- Develop to first order in spherical expansion

$$f(\mathbf{r},\mathbf{p},t) = \frac{1}{4\pi}\left[N(\mathbf{r},p,t) + \frac{3}{v^2}\mathbf{v}\cdot\mathbf{J}(\mathbf{r},p,t)\right]$$

isotropic part — current (dipolar term)

diffusion tensor

$$J_\alpha = -\chi_{\alpha\beta}\nabla_\beta N - \frac{p}{3}\frac{\partial N}{\partial p}u_\alpha$$

Transport equation:

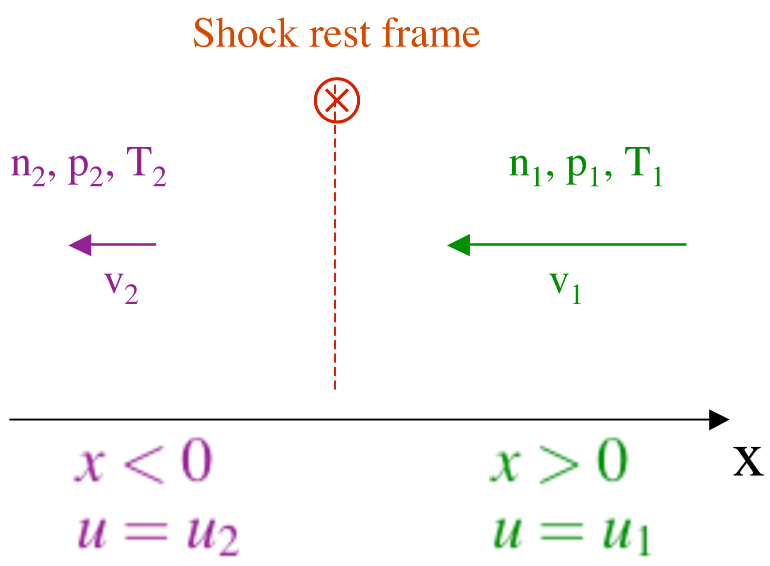
$$\frac{\partial N}{\partial t} + u_\alpha\frac{\partial N}{\partial r_\alpha} - \frac{p}{3}\frac{\partial u_\alpha}{\partial r_\alpha}\frac{\partial N}{\partial p} = \frac{\partial}{\partial r_\alpha}\chi_{\alpha\beta}\frac{\partial N}{\partial r_\beta}$$

Derivation of the shock acceleration spectrum from the transport equation

Just solve the kinetic equation

$$\frac{\partial N}{\partial t} + u_\alpha \frac{\partial N}{\partial r_\alpha} - \frac{p}{3}\frac{\partial u_\alpha}{\partial r_\alpha}\frac{\partial N}{\partial p} = \frac{\partial}{\partial r_\alpha}\chi_{\alpha\beta}\frac{\partial N}{\partial r_\beta}$$

$$\frac{\partial N}{\partial t} + u \frac{\partial N}{\partial x} - \frac{p}{3}\frac{\partial u}{\partial x}\frac{\partial N}{\partial p} = \frac{\partial}{\partial x}\chi\frac{\partial N}{\partial x}$$



- Both upstream and downstream, under the steady state assumption:

$$u\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\chi\frac{\partial N}{\partial x} \quad \Longrightarrow \quad N(x,p) = A(p) + B(p)\exp\left[\int_0^x \frac{u}{\chi(x',p)}\mathrm{d}x'\right]$$

integration constants

$$N(x,p) = A(p) + B(p)\exp\left[\int_0^x \frac{u}{\chi(x',p)}\mathrm{d}x'\right]$$

- **Downstream**: constant particle density

The diffusion coefficient can be anything, but it is > 0 !

$$\Rightarrow \quad N(x,p) = N_0(p)\,; \quad x > 0.$$

- **Upstream**: increase of particle density from infinity up to the shock front:

$$A(p) = N(-\infty,p) \equiv N_{\mathrm{in}}(p)$$

$$N(x,p) = N_{\mathrm{in}}(p) + [N_0(p) - N_{\mathrm{in}}(p)]\exp\left[\int_0^x \frac{u_1}{\chi_1(x',p)}\mathrm{d}x'\right]\,; \quad x < 0$$

- Integrate the transport equation across the discontinuity

$$\frac{\partial N}{\partial t} + u\frac{\partial N}{\partial x} - \frac{p}{3}\underbrace{\frac{\partial u}{\partial x}}_{\propto\ \delta(x)}\frac{\partial N}{\partial p} = \underbrace{\frac{\partial}{\partial x}\chi\frac{\partial N}{\partial x}}$$

$\Delta x \to 0$

$$\Rightarrow \quad \chi_2\frac{\partial N_2}{\partial x} - \chi_1\frac{\partial N_1}{\partial x} = \frac{u_1 - u_2}{3}p\frac{\partial N}{\partial p}; \quad \text{at } x = 0$$

$\chi_2 \frac{\partial N_2}{\partial x}$ estimated at $x = 0^-$; $\chi_1 \frac{\partial N_1}{\partial x}$ estimated at $x = 0^+$

(jump condition for the distribution function)

- Link between far-upstream and far-downstream distribution functions

$$\frac{\mathrm{d}N_{\mathrm{out}}}{\mathrm{d}\ln p} = \frac{3r}{r-1}(N_{\mathrm{in}} - N_{\mathrm{out}})$$

Shock acceleration spectrum

- Final result:

$$N_{\rm out}(p) = \frac{3r}{r-1} p^{-y} \int_0^p \mathrm{d}p' N_{\rm in}(p') p'^{y-1} \quad \text{where} \quad y = \frac{3r}{r-1}$$

- Particles of incoming energy p' are redistributed over a power-law spectrum of index y, which is equal to 4 when r = 4 (strong shock)

- Ex.: monoenergetic injection: $N_{\rm in}(p') = p_0^{-2} N_0 \delta(p' - p_0)$

$$N(E) = \frac{\mathrm{d}n}{\mathrm{d}E} = \frac{\mathrm{d}n}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}E} = \frac{\mathrm{d}p}{\mathrm{d}E} \times p^2 N(p) \Rightarrow N(E) = r(x-1)\frac{N_0}{E_0}\left(\frac{E}{E_0}\right)^{-x}$$

$$\mathrm{d}n = f(\mathbf{r},\mathbf{p},t) \times 4\pi p^2 \mathrm{d}p = p^2 N(\mathbf{r},p,t)\mathrm{d}p$$

where $\boxed{x = \frac{r+2}{r-1}}$

Some limitations

- Injection of particles?
- Isotropy on both sides of the shock?
- Stationarity?
  (astrophysical objects have finite ages!)
- Infinite plane shock wave?
  (astrophysical objects have finite sizes!)

Some comments on new aspects of shock acceleration

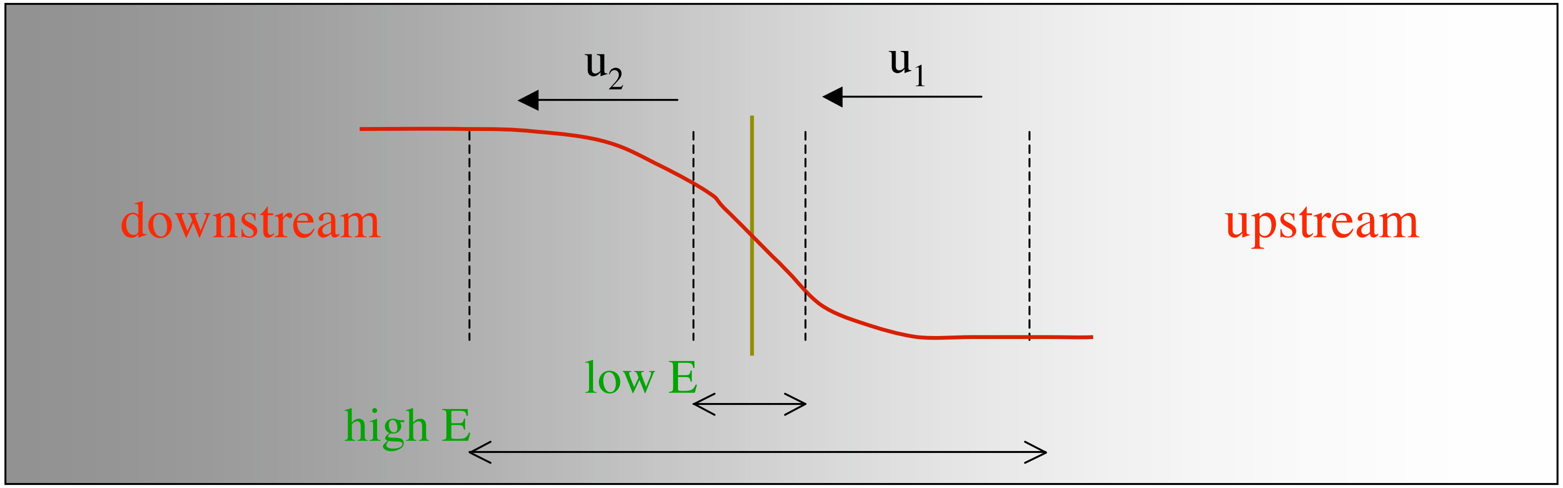
The problem of injection

- Particles must see the shock as a discontinuity (typically a few gyroradii of the thermal protons)
  - Thermal pool → OK
  - But what about electrons?
- Need for injection
  - Internally relativistic plasma (e.g. at the creation of a beam plasma, some dynamo in hot accretion disc models…)
  - Or two-step process: DC fields, acceleration of protons → pions → electrons above 100 MeV, magnetic field lines reconnection, etc.
- Determines the efficiency of shock acceleration

Non-linear shock acceleration

$$N(E)\mathrm{d}E \propto E^{-x}\mathrm{d}E \quad \text{with} \quad x = \frac{r+2}{r-1}$$

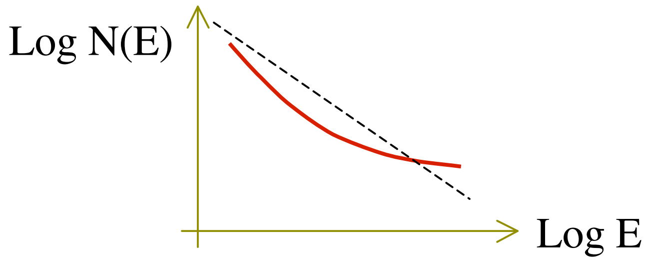
- Modification of the shock structure



• Adiabatic index: $\gamma \rightarrow 5/3$ to $4/3$ $\Rightarrow$ $r \rightarrow 4$ to 7

$\Rightarrow x < 2$

• Effective compression ratio higher for higher energy
$\Rightarrow$ hardening of the spectrum

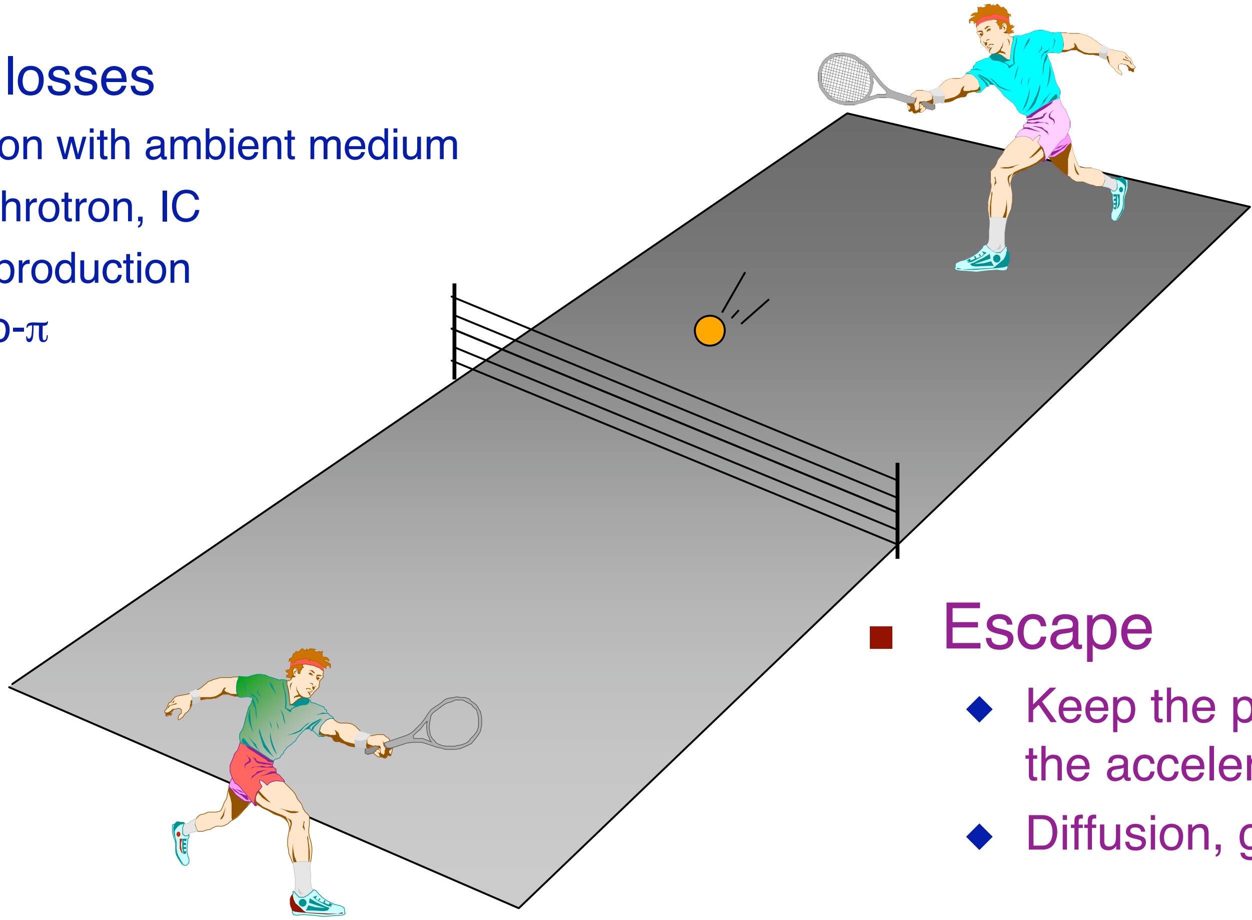


Non linear scheme

- Compression ratio depends on adiabatic index
- But adiabatic index depends on the number of high energy particles: must be set self-consistently
- Moreover, if $r > 4$, $x < 2$: energy dominated by highest energy: infinite if no cutoff!
- → must be an E_{max}, but then leakage of significant energy!
- → not an adiabatic shock anymore: radiation losses, and thus higher compression ratio!
- → fully non-linear situation!

E_{max}: when is the game over?

- Energy losses
  - Friction with ambient medium
  - Synchrotron, IC
  - Pair production
  - photo-π



- Escape
  - Keep the particle in the accelerator
  - Diffusion, gyroradius

- Destruction
  - Photo-disintegration

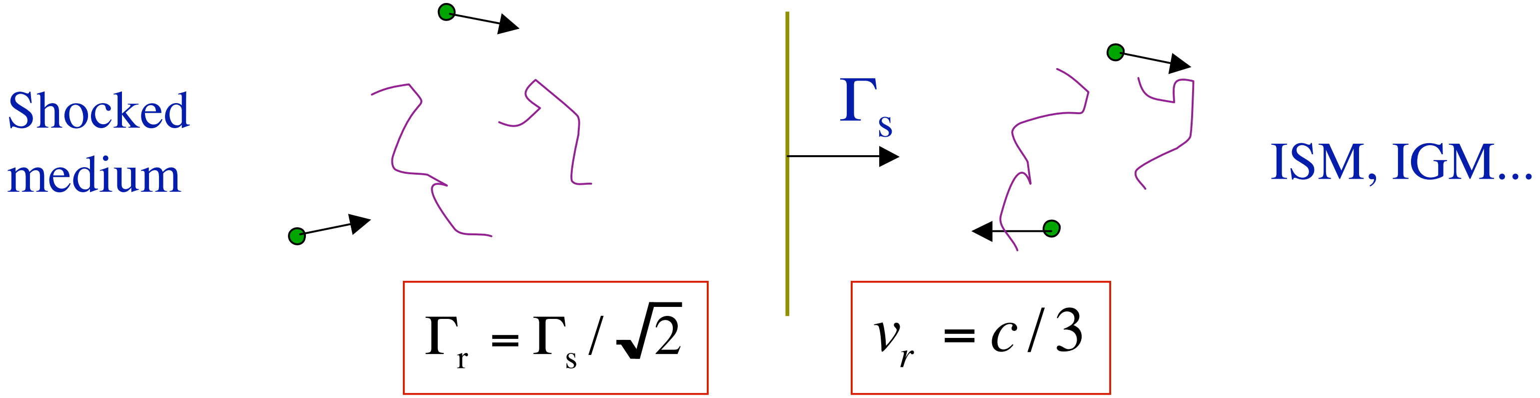
Keep the ball in court!

- Confine the particle within the site: $r_g = E/ZeBc < L$
  - in fact, diffusion-advection at the shock implies:

$$E_{max} \approx Ze \times V_s \times B \times L$$

Acceleration at relativistic shock

- Same principle: acceleration by **change-of-frame**



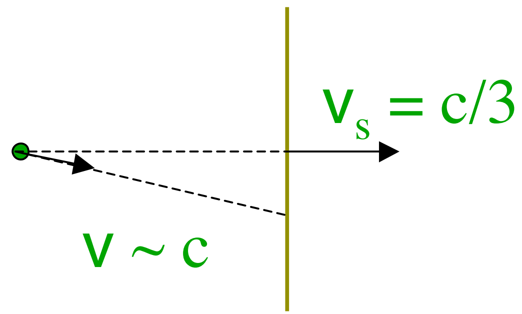
- For the upstream medium, the downstream medium approaches with Lorentz factor Γ_r
  - Energy gain ~ factor Γ_r
- For the downstream medium, the upstream medium approaches with Lorentz factor Γ_r
  - Energy gain ~ an other factor Γ_r
- Finally, $E_f/E_i \sim \Gamma_r^2 \sim \Gamma_s^2 \sim 10^6$!!! 10^9eV → 10^{15} eV→ 10^{21} eV !

Relativistic shock kinematics

$$\frac{E_f}{E_i} = \Gamma_r^2 \left(1 - \beta_r \cos\theta_{\to d}\right)\left(1 + \beta_r \cos\theta'_{\to u}\right) = \frac{E'_f}{E'_i}$$

- Problem: in (ultra-)relativistic shocks, the CR distribution is VERY anisotropic
  - $\Rightarrow$ cos $\theta_{\to d}$ and cos $\theta'_{\to u}$ do not average to ~ zero

Downstream rest frame



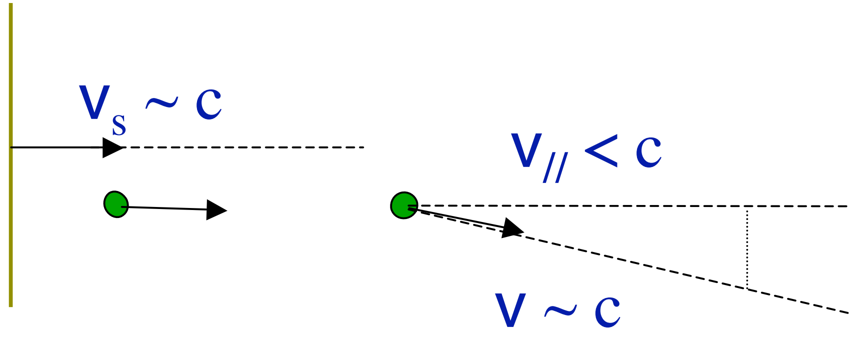
$\cos\theta'_{\to u} > 1/3$

$(\Leftrightarrow \theta_{\to u} < 1/\Gamma_s)$

It's not so easy to overtake the shock, even when one goes at velocity ~ c

$$\frac{E_f}{E_i} = \Gamma_r^2 \left(1 - \beta_r \cos\theta_{\to d}\right)\left(1 + \beta_r \cos\theta'_{\to u}\right) = \frac{E'_f}{E'_i}$$

Upstream rest frame



$\theta_{\to d} < 1/\Gamma_s$

$1 < \Gamma_s\theta_{\to d} \leq 2 \Leftrightarrow -1/3 \leq \cos\theta'_{\to d} < 1/3$

Because the shock is ultra-relativistic (Γ_s), as soon as the particle has been deflected a little, it is overtaken by the shock...

- Energy gain through the up-down change of frame:

$$\frac{E'}{E} = \Gamma_r\left(1 - \beta_r \cos\theta_{\to d}\right) = \Gamma_r\left[1 - \sqrt{1 - 1/\Gamma_r^2} \times \left(1 - 1/2\Gamma_s^2\right)\right] \approx \frac{1}{\Gamma_s} \quad !$$

= energy loss !

Conclusion

- The Γ^2 acceleration works only for the first crossing cycle
- Afterwards, it fails for simple kinematics reasons
  - $E_f/E_i \sim 2$ to 3, depending on the diffusion process
- But it is still very efficient (>> standard shock acc.)
  - E^{-x} , with $x = (1 + \sqrt{13})/2 \sim 2.3$
- Confirmations by numerical simulations and refined Monte-Carlo analysis…

One-cycle boost of GCRs

- Galactic cosmic rays up to the knee: 10^{15} eV
- First cycle across an ultra-relativistic shock (GRB...): $E' = E \times \Gamma^2 \rightarrow 10^{21}$ eV, if $\Gamma \sim 1000$.
- Energetically, **this works** on extra-galactic scale if a reasonable part of the GCRs go through such a shock
- Binary millisecond pulsar systems (PSR 1913+16, 1534+12, 2127+11C: typical GRB progenitors?)
    - Ions with the bulk Lorentz factor of the pulsar wind reach $E_{max} \sim Z \times 10^{20} \times \Gamma_3 \times \ddot{E}_{33}$ eV
    - Power law spectrum $N \propto E^{-2}$ from $Z \times 3.10^{18}$ eV to E_{max}

Summary

- Charged particle interaction with magnetic field can lead to particle acceleration
- Fermi's original idea: transfer energy from macroscopic magnetized structures to individual particles
- These can finally not be "magnetic clouds", but waves and shocks are OK
- Power-laws are naturally produced, and in shock acceleration, the power-law is universal (only depends on shock ratio), with a slope of 2
- Things get more complicated when one goes into the details…