Classical Field Theory

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Electromagnetism: Maxwell's Equations

Maxwell's equations are the basis of electromagnetism. Written in natural rationalized units ($c = 1$ & no 4π factors; a good reference on electromagnetic units is given in the appendix of Jacskon), they are

$$\nabla \cdot \vec{E} = \rho \tag{1}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{2}$$

$$\nabla \cdot \vec{B} = 0 \tag{3}$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j} \tag{4}$$

The first is Coulomb's law; the second Faraday's law of induction; the third forbids magnetic monopoles; and the third is the Ampere-Maxwell law with Maxwell's displacement current. They must be supplemented with the Lorentz force: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$.

These expressions are written in non-covariant form. In manifestly covariant form, the equations become

$$\Box A^\mu = J^\mu \tag{5}$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{6}$$

$$\frac{dP^\alpha}{d\tau} = qU_\beta F^{\alpha\beta} \tag{7}$$

where $A^\mu = (\phi, \vec{A})$ is the 4-potential, $J^\mu = (\rho, \vec{j})$ is the 4-current and with the definition of the *Mawell tensor*: $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. These expressions hold in the *Lorentz gauge* set by the condition $\partial_\alpha A^\alpha = 0$. The first equation results from the inhomogeneous Maxwell equations and tells us that the field propagates as a wave with the speed of light. The second equation is equivalent to the homogeneous Maxwell equations, and the last is the covariant form of the Lorentz force law that also includes energy changes as its time componant.

Electromagnetism: Static Solutions

In the Lorentz gauge we have the wave equation for the 4-potential A^μ with 4-current source J^μ

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) A^\mu = J^\mu \tag{8}$$

Before examining the general solutions, we first study solutions for static fields. This gives us the opportunity to develop some important mathematical tools in a simple situation.

Green Functions and Boundary Conditions

In the case of a static field, the wave equation reduces to the Poisson equation for each of the 4-potential components: $\nabla^2 A^\mu = -J^\mu$. The solution to this equation is well known in infinite space with a confined source distribution

$$A^\mu(\vec{x}) = \frac{1}{4\pi}\int d^3x' \frac{J^\mu(\vec{x}')}{|\vec{x}-\vec{x}'|} \tag{9}$$

We now consider general solutions to the Poisson equation and introduce the idea of Green functions.

A Green function $G(\vec{x},\vec{x}')$ is a solution to

$$\nabla^2 G(\vec{x},\vec{x}') = -\delta(\vec{x}-\vec{x}') \tag{10}$$

from which we see that the formal solution to the Poisson equation is given by

$$A^\mu(\vec{x}) = \int d^3x' G(\vec{x},\vec{x}')J^\mu(\vec{x}') \tag{11}$$

It is left as an exersice to the reader to show that

$$\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\delta(\vec{x}-\vec{x}') \tag{12}$$

i.e., that $(1/4\pi) \times 1/|\vec{x}-\vec{x}'|$ is a Green function for the Laplacian operator; it is the Green function that vanishes at infinity and hence applicable to a finite source distribution in infinite space.

In general, however, we look for solutions over a finite volume subject to prescribed boundary conditions. To study this situation in general, and specifically to identify under which conditions a unique solution exists, we first establish the so-called second Green identities

$$\int_V d^3x \left(\phi\nabla^2\psi + \nabla\psi\cdot\nabla\phi\right) = \int_S dS\left(\phi\frac{\partial\psi}{\partial n}\right) \tag{13}$$

$$\int_V d^3x \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) = \int_S dS\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) \tag{14}$$

where the surface S encloses the volume V and $\partial/\partial n = \hat{n}\cdot\nabla$ is the derivative normal to the surface (towards the exterieur). These relations are easily found by application of the divergence theorem to $\phi\nabla\psi$. Specializing the second identity to $\psi = 1/|\vec{x}-\vec{x}'|$ and $\phi = A^\mu$, we have

$$A^\mu(\vec{x}) = \frac{1}{4\pi}\int_V d^3x' \frac{J^\mu}{|\vec{x}-\vec{x}'|} - \frac{1}{4\pi}\int_S \left(\phi\frac{\partial}{\partial n'}\frac{1}{|\vec{x}-\vec{x}'|} - \frac{1}{|\vec{x}-\vec{x}'|}\frac{\partial A^\mu}{\partial n'}\right) \tag{15}$$

If the surface is S is taken to infinity, where we assume the fields all fall rapidly to zero (for a finite source distribution), then we recover our previous solution.

What conditions must we specify in order to guarantee a unique solution to the Poisson equation? For example, if our boundary conditions fix the field values over a surface bounding the volume of interest, is the sufficient? Consider two possible solutions ϕ_1 and ϕ_2 to the Poisson equation and which both satisfy the same boundary conditions on a given surface. Applying the first Green identity to $\phi = \psi = U \equiv \phi_2 - \phi_1$, which vanishes on the surface, we find

$$\int_V d^3x \left(U\nabla^2U - \nabla U\cdot\nabla U\right) = -\int_V d^3x|\nabla U|^2 = 0 \tag{16}$$

since $\nabla^2 U = 0$ in the volume. The conclusion is that $\nabla U = 0$ in the volume, which implies $U = 0$, because $U = 0$on the bounding surface, and hence $\phi_2 = \phi_1$ in the volume; in other words, specifying the field values over a bounding surface uniquely identifies the solution to the Poisson equation in the bounded volume. This kind of boundary conditions are known as *Dirichlet conditions.*

It is left as an exercise to show that *Neumann boundary conditions*, where normal derivative of the field on the surface is given, also leads to a unique solution, to within an additive constant.

Finally, note that the Green function for the Poisson equation is in general

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}') \tag{17}$$

where the function F is a solution to the Laplace equation $\nabla^2 F(\vec{x},\vec{x}') = 0$. The Green function appropriate for a given problem depends on the boundary conditions, which then adapt the function F, and hence G, to the problem of interest.

Orthogonal Functions

Orthogonal functions offer a powerful and general method for solving field theory (differential equation) problems. In the present case of static solutions to the wave equation (i.e., the Poisson equation) we are interested in the Laplacian operator. Consider a slight generalization of this operator $D \equiv \nabla^2 + f(\vec{x})$, with f a real function. Note first that D is then a *hermitian operator*:

$$\int dV f D g = \int dV g D f \tag{18}$$

Now consider solutions to the *eigenvalue problem* associated with this hermitian operator

$$D\psi_\lambda = \lambda\psi_\lambda \tag{19}$$

where λ is the *eigenvalue* corresponding to the *eigenfunction* ψ_λ. In general there are only solutions for specific values of λ, although continous eignvalues are also possible.

The eigenvalues and eigenfunctions of an hermitian operator have three essential properties:

1. The eigenvalues are real: $\lambda^* = \lambda$
2. The eigenfunctions are orthonomal: $dV\psi_n^*\psi_m = \delta_{nm}$ for the discrete case, and $\int dV\psi_\lambda^*\psi_{\lambda'} = \delta(\lambda-\lambda')$ for the continous case
3. They form a complete basis in the space of solutions to differential equations involving the operator D.

The first 2 properties are straightforward to prove, while the third takes more consideration.

Complete sets of orthogonal functions are extremely useful. We can use them, for example, to find the Green function for a differential equation with specified boundary conditions. For instance, consider expanding the the Green function for the Laplacian in infinite space in terms of Fourier modes $\psi_{\vec{k}}(\vec{x}) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}}$. We seek the solution to $\nabla^2 G(\vec{x},\vec{x}') = -\delta(\vec{x}-\vec{x}')$, and we know that we can expand $G(\vec{x},\vec{x}') = \int d^3k G_{\vec{k}}(\vec{x}')\psi_{\vec{k}}(\vec{x})$. Furthermore, $\delta(\vec{x}-\vec{x}') = \int d^3k \psi_{\vec{k}}^*(\vec{x}')\psi_{\vec{k}}(\vec{x})$. Substituting into the previous differential equation, we find

$$G_{\vec{k}}(\vec{x}') = \frac{1}{k^2}\psi_{\vec{k}}^*(\vec{x}') = \frac{1}{(2\pi)^{3/2}}\frac{e^{-i\vec{k}\cdot\vec{x}'}}{k^2} \tag{20}$$

the real-space expression for our Green function is the integral of this expression over k-space

$$G(\vec{x},\vec{x}') = \int d^3k \frac{1}{(2\pi)^3}\frac{e^{-i\vec{k}\cdot\vec{x}'}}{k^2} e^{i\vec{k}\cdot\vec{x}} \tag{21}$$

In fact, we see clearly that $G(\vec{x},\vec{x}') = G(\vec{x}-\vec{x}')$ and so consider the integral

$$G(\vec{r}) = \frac{1}{(2\pi)^3}\int d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2} = \frac{1}{(2\pi)^2}\int_0 dk \frac{(e^{ikr}-e^{-ikr})}{ikr} = \frac{1}{(2\pi)^2}\frac{1}{r}\frac{1}{2}\int_{-\infty}^{\infty} dx \frac{(e^{ix}-e^{-ix})}{ix} \tag{22}$$

which can be evaluated as a closed contour in the complex x-plane. The result is $G(\vec{r}) = \frac{1}{4\pi}\frac{1}{r}$, or in other words

$$G(\vec{x} - \vec{x}') = \frac{1}{4\pi}\frac{1}{|\vec{x} - \vec{x}'|} \tag{23}$$

which we already knew.

Fourier eigenfunctions are only one example. When dealing with spherical symmetry, which also applies to infinite space, we turn as usual to spherical harmonics. In spherical coordinates the Laplacian takes the form

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{L^2}{(\hbar r)^2} \tag{24}$$

where, for convenience, we employ the operator notation from quantum mechanics

$$\frac{L^2}{(\hbar r)^2} = -\frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{\partial^2}{\partial\phi^2}\right] \tag{25}$$

The spherical harmonics are eigenfunctions of this angular operator: $L^2 Y_{lm}(\theta,\phi) = \hbar^2 l(l+1)Y_{lm}$. They form a complete basis on the unit sphere, so that we can always decompose an arbitrary function as $f(r,\theta,\phi) = \sum_{lm} a_{lm}(r)Y_{lm}(\theta,\phi)$.

To find the expansion for the Green function in this basis, we first develop the expression for the delta function, which in spherical coordinates takes the form $\delta(\vec{r}-\vec{r}') = \frac{1}{r^2}\delta(r-r')\delta(\cos\theta-\cos\theta')\delta(\phi-\phi')$. Treating this as a function of $\vec{r}$, we expand as above to find $a_{lm}(r) = \frac{1}{r^2}\delta(r-r')Y^*_{lm}(\theta',\phi')$. We also expand the Green function $G(\vec{r},\vec{r}') = \sum_{lm} A_{lm}(r|\vec{r}')Y_{lm}(\theta,\phi)$. Substituting into the differential equation defining the Green function, we find $A_{lm} = g_l(r,r')Y^*_{lm}(\theta',\phi')$ where

$$\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{l(l+1)}{r^2}\right)g_l(r,r') = -\frac{1}{r^2}\delta(r-r') \tag{26}$$

Now, when $r \neq r'$the solution is $g_l(r,r') = A(r')r^l + B(r')r^{-(l+1)}$. The coefficients are set by the boundary conditions, the delta function in the above equation and the symmetry imposed on g_l :$g_l(r,r') = g_l(r',r)$. For infinite space (to avoid singularities), we must have

$$g_l(r,r') = Ar^l \tag{27}$$
$$= Br^{-(l+1)} \tag{28}$$

the first line for $r < r'$and the second for $r > r'$. The exchange symmetry implies that we can in fact write

$$g_l(r,r') = C\frac{r_<^l}{r_>^{l+1}} \tag{29}$$

where $r_<(r_>)$ is the smaller (greater) of r and r'. Finally, the constant C is fixed by the delta function by integrating the differential equation in r:

$$\frac{d}{dr}rg_l(r,r')|_{r=r'+\epsilon} - \frac{d}{dr}rg_l(r,r')\,|_{r=r'-\epsilon} = -\frac{1}{r'} \tag{30}$$

which, when applied to our above expression, yields $C = \frac{1}{2l+1}$. In other words,

$$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} = \sum_{lm}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\phi')Y(\theta,\phi) \tag{31}$$

Multipole Expansion

The above spherical expansion for the Green function allows us to introduce the multipole expansion for static solutions. We develop the general Green function solution to the field as

$$A^\mu(\vec{r}) = \sum_{lm} \frac{1}{2l+1} q^\mu_{lm} \frac{1}{r^{l+1}} Y_{lm}(\theta, \phi) \tag{32}$$

Using the above spherical expansion for the Green function in the general solution at a point outside of the source distribtution, we find

$$q^\mu_{lm} = \int d^3r' r'^l Y^*_{lm}(\theta', \phi') J^\mu(\vec{r}') \tag{33}$$

which are known as the multipole moments:

$$q_{00} = \frac{1}{\sqrt{4\pi}} \int d^3r' J^\mu = \frac{1}{\sqrt{4\pi}} Q \tag{34}$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}} \int d^3r'(x' - iy') J^\mu = -\sqrt{\frac{3}{8\pi}} (P_x - iP_y) \tag{35}$$

$$q_{10} = \sqrt{\frac{3}{4\pi}} \int d^3r' z' J^\mu = \sqrt{\frac{3}{4\pi}} P_z \tag{36}$$

etc. Note that $q_{l,-m} = (-1)^m q^*_{lm}$. This allows us to write

$$A^\mu = \frac{Q}{4\pi r} + \frac{\vec{P} \cdot \vec{r}}{r^3} + ... \tag{37}$$

where Q is the total "charge" of the source J^μ and $\vec{P}$ is its dipole moment. Note that successive terms fall off with higher powers of $1/r$; thus, at large distances, only the first few terms matter, i.e., as we move away from the source distribution, eventually field tends towards a monopole field, because the source appears as a point distribution.

Electromagnetism: The Wave Equation

Thus far in our studies, we have focused on the static case. This has permitted us to introduce a number of important concepts and techniques in a somewhat familiar setting and without the added complication of propagation effects. We now consider general solutions to the electromagnetic field equations in this section, where we will apply the techniques we've been using in the static case to find solutions to the wave equation.

Solutions to the Wave Equation: Green Functions

General solutions to the electromagnetic field obey the wave equation for the 4-potential

$$\Box A^\mu(\mathbf{x}) = J^\mu(\mathbf{x}) \tag{38}$$

in the Lorentz gauge $\partial_\mu A^\mu = 0$. Hereafter, we denote 4-vectors with boldfaced lettering; for example, $\mathbf{x}$ denotes the position 4-vector above. Following the same approach as for the Poisson equation, we seek a Green function for this differential equation

$$\Box G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \tag{39}$$

and corresponding boundary conditions. Note there is no minus sign in this equation. With the Green function, we write the general solution as a 4-dimensional integral over the source distribution

$$A^\mu(\mathbf{x}) = \int d^4x' G(\mathbf{x},\mathbf{x}')J^\mu(\mathbf{x}') \tag{40}$$

We shall be primarily interested in the case of infinite space with appropriate boundary conditions at infinity. As before, we first apply Fourier techniques to find the Green function, and then turn to spherical expansions of the Green function to derive a multipole expansion for radiating sources.

Fourier Representation

The 4-dimensional Fourier representation for the Green function is

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{(2\pi)^2}\int d^4k G(\mathbf{k},\mathbf{x}')e^{i\mathbf{k}\cdot\mathbf{x}} \tag{41}$$

Applying to the wave equation, we find

$$G(\mathbf{k},\mathbf{x}') = -\frac{1}{(2\pi)^2}\frac{e^{-i\mathbf{k}\cdot\mathbf{x}'}}{\mathbf{k}^2} \tag{42}$$

where we emphasize that $\mathbf{k}^2 = (k^0)^2 - \vec{k}^2$.

Once again, we use contour integration - this time in the complex k^0-plane - to integrate to obtain the expression in real space. The above function has two poles along the real axis, at $k^0 = \pm\vec{k}^2 = \pm k^2$. Consider first the case where $x^0 > x'^0$, and we may close the contour in the upper half plane:

$$\int_{-\infty}^{\infty}\frac{e^{ik^0\cdot(x^0-x'^0)}}{(k^0)^2-k^2} = \frac{i\pi}{k}\left[e^{ik(x^0-x'^0)} - e^{-ik(x^0-x'^0)}\right] \tag{43}$$

Replacing this result in the Fourier integral and integrating over the direction of $\vec{k}$, i.e., over $d\Omega_{\vec{k}}$, we obtain

$$G(\mathbf{x},\mathbf{x}') = -\frac{1}{8\pi^2}\frac{1}{R}\int_0^\infty dk(e^{ik\tau} - e^{-ik\tau})(e^{ikR} - e^{-ikR})\Theta(\tau) \tag{44}$$

where $\tau \equiv x^0 - x'^0$and $R \equiv |\vec{x} - \vec{x}'|$, and the Heaviside function indicates that the result applies to $\tau > 0$. The integrand being pair, we can extend the integral from $-\infty$ to ∞, dividing by 2. The result yields Dirac delta functions and the final expression

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi}\frac{\delta\left[(x^0-x'^0) - |\vec{x}-\vec{x}'|\right]}{|\vec{x}-\vec{x}'|}\Theta(x^0-x'^0) \tag{45}$$

This is known as the *retarded Green function*; it displays the characteristic $1/R$ behaviour of the electromagnetic potential and a time delay for effects at the source to propagate with the speed of light ($c = 1$) to the point at which the field is being evaluated. Specifically, an change at position $\vec{x}'$ in the source occuring at time x'^0 produces an effect in the field at position $\vec{x}$ at time $x^0 = x'^0 + |\vec{x} - \vec{x}'|$, as expected.

It is left as an exercise to the reader to perform the calculation for the case $x^0 < x'^0$and to interpret the result. The Green function in this case is known as the *advanced Green function.*

Spherical Expansion

The spherical expansion in fact only concerns the spatial part of the Green function. Hence, we first develop the temporal part with a Fourier analysis

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk e^{-ikx^0} G(\vec{x},k,\mathbf{x}') \tag{46}$$

where in fact $k = k^0$(we drop the superscript for simplicity). Note that the development is over $\mathbf{x}$ and we treat, as before, $\mathbf{x}'$as a parameter. The delta function

$$\delta(\mathbf{x} - \mathbf{x}') = \delta(x^0 - x'^0)\delta(\vec{x} - \vec{x}') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(x^0 - x'^0)}\delta(\vec{x} - \vec{x}') \tag{47}$$

Substitution into the equation for the Green function, we find

$$-(\nabla^2 + k^2)G(\vec{x}, k, \mathbf{x}') = \frac{e^{ikx'^0}}{\sqrt{2\pi}}\delta(\vec{x} - \vec{x}') \tag{48}$$

suggesting that we separate out the x'^0dependence as $G(\vec{x}, k, \mathbf{x}') = g(\vec{x}, k, \vec{x}')e^{ikx'^0}/\sqrt{2\pi}$. This gives us the equation

$$(\nabla^2 + k^2)g(\vec{x}, k, \vec{x}') = -\delta(\vec{x} - \vec{x}') \tag{49}$$

We have thus eliminated all time dependence and may proceed, as before, by expanding our spatial functions in spherical harmonics. From our previous experience, we immediately pose

$$g(\vec{x}, k, \vec{x}') = \sum_{lm} a_l(r, k, r')Y^*_{lm}(\theta', \phi')Y_{lm}(\theta, \phi) \tag{50}$$

When put into the above differential equation, with the delta function expressed in spherical coordinates (see above), we find

$$\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{l(l+1)}{r^2} + k^2\right)a_l(r, k, r') = -\frac{1}{r^2}\delta(r - r') \tag{51}$$

or

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2\right)a_l(r, k, r') = -\frac{1}{r^2}\delta(r - r') \tag{52}$$

For $r \neq r'$ (i.e., the homogeneous equation), the solutions to this differential equation are linear combinations of *spherical Bessel functions*, $j_l(kr)$and $n_l(kr)$. The behavior of these functions at small and large arguments are given by

$$j_l(x) \rightarrow \frac{x^l}{(2l+1)!!} \tag{53}$$

$$n_l(x) \rightarrow -\frac{(2l-1)!!}{x^{l+1}} \tag{54}$$

for $x << 1$ and $x << l$; and

$$j_l(x) \rightarrow \frac{1}{x}\sin(x - \frac{l\pi}{2}) \tag{55}$$

$$n_l(x) \rightarrow -\frac{1}{x}\cos(x - \frac{l\pi}{2}) \tag{56}$$

at $x >> l$. In these expressions, $x!! \equiv x(x-2)(x-4)...$ We also define the *Hankel functions*

$$h_l^{(1,2)}(x) \equiv j_l(x) \pm i n_l(x) \tag{57}$$

whose behavior at large x is therefore

$$h_l^{(1,2)}(x) \rightarrow (\mp i)^{l+1}\frac{e^{\pm ix}}{x}. \tag{58}$$

If we seek the *outgoing Green function* (i.e., describing waves propagating outward from a source - the retarded Green function), then the solution with the proper boundary conditions is

$$a_l(r, k, r') = A_l j(kr_<)h^{(1)}(kr_>) \tag{59}$$

since it does not diverge at small radius and represents an outgoing plane wave at large radius. The coefficient is determined by the discontinuity imposed by the delta function at $r = r'$. The result is (exercise) $A_l(k) = ik$.

Putting all this back together, we have our spherical expansion

$$g_l(\vec{x}, k, \vec{x}') = ik \sum_{lm} j_l(kr_<) h_l^{(1)}(kr_>) Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \tag{60}$$

and

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} dk g_l(\vec{x}, k, \vec{x}') e^{-ik(x^0 - x'^0)} \tag{61}$$

Multipole Expansion

We now use the previous result to obtain the multipole expansion for the wave equation. We consider, as before, a localized source distribution $J^\mu(\mathbf{x}')$ and seek to calculate the field $A^\mu(\mathbf{x})$ outside the source. Using our spherical expansion in the general solution provided by the Green function, we have

$$A^\mu(\mathbf{x}) = \int d^4x' G(\mathbf{x}, \mathbf{x}') J^\mu(\mathbf{x}') \tag{62}$$

$$= \int d^3x' \int dk \frac{e^{ikx^0}}{\sqrt{2\pi}} g_l(\vec{x}, k, \vec{x}') J^\mu(\vec{x}', k) \tag{63}$$

where $J^\mu(\mathbf{x}) = (1/\sqrt{2\pi}) \int dk J^\mu(\vec{x}, k) e^{-ikx^0}$. If we define similarly $A^\mu(\mathbf{x}) = (1/\sqrt{2\pi}) \int dk A^\mu(\vec{x}, k) e^{-ikx^0}$, we can write

$$A^\mu(\vec{x}, k) = \int d^3x' g_l(\vec{x}, k, \vec{x}') J^\mu(\vec{x}', k) \tag{64}$$

$$= ik \sum_{lm} \left[\int d^3x' j_l(kr') Y_{lm}^*(\theta', \phi') J^\mu(\vec{x}', k) \right] h_l^{(1)}(kr) Y_{lm}(\theta, \phi) \tag{65}$$

In the *far field*, i.e., $kr >> 1 >> kr'$ we then have

$$A^\mu(\vec{x}, k) \to ik \sum a_{lm}(k) (-i)^{l+1} \frac{e^{ikr}}{kr} Y_{lm}(\theta, \phi) \tag{66}$$

with

$$a_{lm}(k) = \int d^3x' \frac{x^l}{(2l+1)!!} Y_{lm}^*(\theta', \phi') J^\mu(\vec{x}', k) \tag{67}$$

Putting it all together, we see

$$A^\mu(\mathbf{x}) \to \frac{i}{\sqrt{2\pi}} \int dk k \sum_{lm} a_{lm}(k) (-i)^{l+1} Y_{lm}(\theta, \phi) \frac{e^{-ik(x_0 - r)}}{kr} \tag{68}$$

Poynting's Theorem

The theorem expresses the conservation of energy and momentum in presence of the electromagnetic field. Consider a volume enclosing a finite distribution of charge and current. Within this volume, the quantity $\vec{j} \cdot \vec{E}$ represents the power exerted on the distribution by the electromagnetic field (only the electric field does work according the Lorentz law). We may eliminate the electric current in favor of the field by using Maxwell's equations to find

$$\vec{j} \cdot \vec{E} = \vec{E} \cdot \nabla \times \vec{B} - \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \tag{69}$$

Employing the identity $\nabla \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{B}$ and Faraday-Maxwell equation, we obtain

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{S} = -\vec{j} \cdot \vec{E} \tag{70}$$

where $u \equiv (1/2)(E^2 + B^2)$ is the energy density of the field, $\vec{S} \equiv \vec{E} \times \vec{B}$ is the *Poynting vector*. This is a conservation equation telling us that any change in the total energy, field plus mechanical energy of the sources, is accounted for by the flow of energy across the surface bounding the V as expressed by the Poynting vector. The latter is therefore the energy flux of the field.

We may pursue the same kind of argument for the 3-momentum. The change in momentum of the charge distribution within the volume V is

$$\frac{d\vec{P}}{dt} = \int dV(\rho\vec{E} + \vec{j} \times \vec{B}) \tag{71}$$

Using the inhomogeneous Maxwell equations to eliminate the source quantities, we find

$$\frac{d\vec{P}}{dt} + \frac{d}{dt}\int dV(\vec{E} \times \vec{B}) = \int dV \left[\vec{E}(\nabla \cdot \vec{E}) - \vec{E} \times (\nabla \times \vec{E}) + \vec{B} \cdot (\nabla \cdot \vec{B}) - \vec{B} \times (\nabla \times \vec{B})\right] \tag{72}$$

where we also used the fact that $\nabla \cdot \vec{B} = 0$. Noting that

$$\left[\vec{E}(\nabla \cdot \vec{E}) - \vec{E} \times (\nabla \times \vec{E})\right]_i = \sum_j \partial_j (E_i E_j - \frac{1}{2}\delta_{ij} E^2) \tag{73}$$

we can write

$$\frac{d}{dt}(\vec{P}_{sources} + \vec{P}_{field})_i = \sum \int dV \partial_j T_{ij} = \int_{surface} dS \sum_j T_{ij} n_j \tag{74}$$

where $\hat{n}$ is the normal to the surface enclosing the volume. Here, we define $\vec{P}_{field} \equiv \vec{S} = \vec{E} \times \vec{B}$. This equation tells us that the total momentum of the sources plus the field is conserved, any changes in the quantity within the volume being accounted for by the flow of momentum across the surface as expressed by the field momentum tensor $T_{ij} \equiv E_i E_j + B_i B_j - u\delta_{ij}$.

Klein-Gordon Equation

The Schrodinger equation can be obtained by the correspondence principle from the Newtonian energy equation, $E = \frac{p^2}{2m} + V(\vec{r})$, by associating $E = i\partial_t$ and $\vec{p} = -i\nabla$ (in units where $\hbar = 1$). It is therefore a non-relativistic quantum mechanical wave equation. When looking for a relativistic generalization, it is natural to first try using the expression $P^\mu P_\mu = m^2$, which leads to the Klein-Gordon equation for a field ϕ:

$$(\Box + m^2)\phi = 0 \tag{75}$$

It is, however, not possible to use this new field ϕ as a probability amplitude. Recall that in the case of the Schrodinger equation, the *probability density* is defined as $\rho = |\psi|^2$ because this quantity is conserved: $\dot{\rho} + \nabla \cdot \vec{j} = 0$ where the *probability current* is $\vec{j} = (1/2mi)(\psi^* \nabla\psi - \psi\nabla\psi^*)$. In the non-relativistic theory, this makes perfect sense, since particles are neither created or destroyed; the probability density is positive definite and conserved.

In the case of the Klein-Gordon equation, there is no conserved, *positive-definite* quantity. There is, however, a conserved 4-current, $\partial_\mu J^\mu = 0$ with

$$J_\mu \equiv \frac{i}{2m}(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*) \tag{76}$$

The conserved 4-current can be interpreted in terms of a generalized charge, but we cannot take $\rho = J^0$ as a probability density because it takes on negative values. This fact lead Dirac to search for an alternate relativistic generalization that we will explore below.

General solutions to the Klein-Gordon equation are written in terms of Fourier intergrals as

$$\phi(\mathbf{x}) = \frac{1}{(2\pi)^2} \int d^4p\phi(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} \tag{77}$$

where here $\mathbf{p}$ and $\mathbf{x}$ refer to 4-vectors (note that in our natural units, $\mathbf{p} = \mathbf{k}$). The Klein-Gordon equation imposes the relation $p^0 = E = \pm\sqrt{\vec{p}^2 + m^2} \equiv \pm\omega_p$, implying the existence of negative energy solutions that present a problem of interpretation.

Consider now the propagator for this classical scalar field, which we find by solving $(\Box + m^2)G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$. Using the Fourier representation and our now standard approach, we find

$$G(\mathbf{k}, \mathbf{x}') = -\frac{1}{(2\pi)^2} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}'}}{\mathbf{k}^2 - m^2} \tag{78}$$

Dirac Equation

The search for a relativistic generalization of the Schrodinger equation lead Dirac to propose an alternative to the Klein-Gordon equation. The primary motivation was to find an equation with a conserved positive definite quantity. Such a quantity exists in quantum mechanics because the Schrodinger equation is first order in the time derivative (recall basic quantum mechanics). Since in relativity we seek to place time and space on equal footing, Dirac proposed an equation that is first order in both time and space derivatives:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{79}$$

where the objects γ^μ satisfy must satisfy *anticommutation relations*

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{80}$$

where $\eta^{\mu\nu}$is the usual Minkowski metric. This guarantees that

$$(i\gamma^\mu \partial_\mu - m)(i\gamma^\mu \partial_\mu - m)\psi = (\Box + m^2)\psi = 0 \tag{81}$$

i.e., the field also satisfies the Klein-Gordon equation; in other words, we recover the fundamental relation $\mathbf{p}^2 = m^2$, as we must for a relativistic particle.

What are the γobjects? We can represent them with matrices satisfying the above anticommutation relations. The lowest possible dimension is 4; they then become 4x4 matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{82}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \tag{83}$$

where I refers to the 2x2 unit matrix and σ^i refer to the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{84}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{85}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{86}$$

The important conclusion of this is that the field $\psi$is in fact a column vector with 4 components, known as a *spinor.* In other words, there are actually 4 fields making up ψ.

To find the conserved current, we must first introduce the complex conjugate Dirac equation, or the *adjoint Dirac equation*, since we are dealing with matricies and spinors:

$$i\left(\partial_\mu\psi^\dagger\right)\gamma^{\dagger\mu}+m\psi^\dagger=\psi^\dagger(i\gamma^{\dagger\mu}\overleftarrow{\partial}_\mu+m)=0 \tag{87}$$

Note that the order is important and the over-arrow indicates that we take the derivative on objects to the left. Given the above expressions for the gamma matricies, we see that $\gamma^{0\dagger}=\gamma^0$ and $\gamma^{i\dagger}=\gamma^0\gamma^i\gamma^0$. By introducing $\bar{\psi}\equiv\psi^\dagger\gamma^0$ we can write the adjoint equation as

$$\bar{\psi}\left(i\gamma^\mu\overleftarrow{\partial}_\mu+m\right)=0 \tag{88}$$

The conserved 4-current is then $J^\mu\equiv\bar{\psi}\gamma^\mu\psi$, i.e., the Dirac equation guarantees that $\partial_\mu J^\mu=0$. Note that $J^0=\rho$ is indeed a positive quantity.

Hamilton-Lagrange Approach

In this section we develop the powerful approach based on Hamilton's principle of least action. We begin with a brief review of its application in classical mechanics before turning to field theory.

Review of Application to Classical Mechanics

Newton's laws summarize classical mechanics in terms of the action of *forces.* Hamilton proposed an alternate foundation based on the principle of least action. The two approaches to classical mechanics are equivalent, but Hamilton's approach has proved the most useful for generalization to field theory and to quantum mechanics. It also brings more directly to light the relation between symmetries and dynamics. We begin here by reviewing Hamilton-Lagrange dynamics applied to point particle systems.

Hamilton's least-action principle states that of all the possible paths $q(t)$ between two points,(t_1,x_1) & (t_2,x_2), a particle of mass m will take that one that minimizes the *action*

$$S[q(t)]=\int_{t_1}^{t_2}dtL[q(t),\dot{q}(t),t] \tag{89}$$

where $q(t_1)=x_1$& $q(t_2)=x_2$, and the function L is known as the *Lagrangian.* Note that S associates a number (the value of the action) to each possible path; it is hence called a *functional.* Leaving aside the all-important question of how to construct the Lagrangian for the moment, we seek first the path $q_c(t)$ that minimizes the action (the one actually taken by the classical particle). To this end we consider an arbitrary path $q(t)=q_c(t)+\delta q(t)$ expressed in terms of a variation $\delta q(t)$ away from the classical path $q_c(t)$. The action is, by definition, stationary for q_c, so that for any small variation we must have

$$\delta S=0=\int_{t_1}^{t_2}dt\left\{\delta q\frac{\delta L}{\delta q}+\delta\dot{q}\frac{\delta L}{\delta\dot{q}}\right\} \tag{90}$$

Since $\delta\dot{q}=d\delta q/dt$ we can integrate the last term by parts to get

$$0=\int_{t_1}^{t_2}dt\left\{\frac{\delta L}{\delta q}-\frac{d}{dt}\frac{\delta L}{\delta\dot{q}}\right\}\delta q \tag{91}$$

where we use the fact that the surface terms are zero because by definition the variation vanishes at these fixed points: $\delta q(t_{1,2})=0$. This integral equation holds for any variation function δq, which yields the *Lagrange equations of motion:*

$$\frac{\delta L}{\delta q_i}-\frac{d}{dt}\frac{\delta L}{\delta\dot{q}_i}=0 \tag{92}$$

where here we generalize to several coordinates q_i describing the path; these can be, for example, the three coordinates of a particle in 3D space, or the coordinates of a system of several particles.

For simple systems, the Lagangian is $L = T - V$, i.e., the kinetic energy minus the potential energy of the system. For example, we readily see that this leads to Newton's equations for a single point mass in one dimension subject to a conservative force: $L = (1/2)m\dot{x}^2 - V(x)$ and

$$-\frac{dV}{dx} - m\ddot{x} = 0 \tag{93}$$

in other words, $m\ddot{x} = F$.

We may also formulate mechanics in terms of the *Hamiltonian,* instead of the Lagrangian. This is an approach of extreme utility for quantum mechanics and quantum field theory. The Hamiltonian is defined as $H(q,p) = p\dot{q} - L(q,\dot{q})$, and taken to be a function of the coordinate q and of the *generalized momentum* $p \equiv \partial L/\partial \dot{q}$ (recall the case of a single point mass subjected to a conservative force). Taking the differential and using the Lagrange equations of motion, we find

$$dH = \dot{q}dp + pd\dot{q} - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial \dot{q}}d\dot{q} = \dot{q}dp - \dot{p}dq \tag{94}$$

Equating this result to the general expresion $dH = \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp$, we find the *canonical equations of motion (or the Hamilton equations):*

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{95}$$

$$-\dot{p}_i = \frac{\partial H}{\partial q_i} \tag{96}$$

Comments on the importance/reality of energy, in particular potential energy. Scalar quantities. Conservation laws and symmetries of H and L.

Application to Fields

For simplicity we first consider a scalar field $\phi(\mathbf{x})$. The field is a system with an infinite number of degrees-of-freedom, namely the field value at each point in space. In classical mechanics, the Lagrangian is the difference of the *total* kinetic and potential energies of a system. For the case of a simple system of point masses, this means that the Lagrangian is the sum of the Lagrangians that we could associate with each particle. Taking this as insperation, we seek to construct a Lagrangian for the field that sums over the contribution from each point in space. Hence, we define the *Lagrange density* $\mathcal{L}(\phi, \partial_\mu\phi)$ as a function of the field and its space-time derivatives at each point, and the action as

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu\phi) \tag{97}$$

Since the 4-volume element d^4x is an invariant under Lorentz Transformations, the Lagrange density is a Lorentz scalar. Following the same line of argument as in classical mechanics, we minimize the action to find the dynamical field equations:

$$\frac{\delta\mathcal{L}}{\delta\phi(\mathbf{x})} - \partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} = 0 \tag{98}$$

The Klein-Gordon Field

The Lagragian for a real Klein-Gordon field is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{99}$$

We see that the first term is quadratic in first-order derivatives of the field (our generalized coordinates); we therefore refer to the first term as the *kinetic energy* of the field, while the second term represents a *potential energy* associated with the field. Apply the Lagrange equations to this expression, we find

$$\partial_\mu \partial^\mu \phi + m^2 \phi = (\Box + m^2)\phi = 0 \tag{100}$$

as required for the KG field. For a complex KG field, we have two independent degrees-of-freedom at each point, represented by ϕ and ϕ^*. In this case the Lagrangian is

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 |\phi|^2 \tag{101}$$

Dirac Field

The Lagrangian is given by

$$\mathcal{L} = -\bar{\psi} \left[\frac{i}{2} \left(\gamma^\mu \overleftarrow{\partial}_\mu - \gamma^\mu \partial_\mu \right) + m \right] \psi \tag{102}$$

Electromagnetic Field

The Lagrangian is constructed from both the Maxwell tensor and the 4-potential

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j^\mu A_\mu \tag{103}$$

The Lagrangian for the *free* field is obtained when $j^\mu = 0$.

Symmetries and Conservated Currents

The Lagrangian method brings to light a key relation between symmetries and conserved quantities. We now explore this relation, generally known as *Noether's theorem*: *to every continuous one-parameter set of invariances (symmetries) of a system there is an associated locally conserved current.*

An important example is the following. Let $\mathcal{L}[\phi, \partial_\mu \phi]$ be the Lagrangian for a field ϕ that depends on the space-time coordinates only through the dependence of the field and its derivatives (usually the case), and consider a translation of the system, $x^\mu \to x'^\mu = x^\mu + a^\mu(\mathbf{x})$. Note that the translation depends in general on the space-time position. The changes to the field and its derivatives are

$$\delta\phi = \delta a^\mu \partial_\mu \phi \tag{104}$$

$$\delta \partial_\mu \phi = \delta a^\nu \partial_\mu \partial_\nu \phi + [\partial_\mu \delta a^\nu] \partial_\nu \phi \tag{105}$$

leading to a variation in the action of

$$\delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\delta \partial_\mu \phi} \delta \partial_\mu \phi \right\} \tag{106}$$

$$= \int d^4x \left\{ \partial_\nu \mathcal{L} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi \right] \right\} \delta a^\nu \tag{107}$$

where we have integrated by parts to obtain the second line. If the system is invariant to this change, i.e., has a symmetry with respect to space-time translations, then $\delta S = 0$ for arbitrary δa^ν. The implication is then that $\partial_\mu \Theta^{\mu\nu} = 0$ with the definition

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} \tag{108}$$

To understand this object, let's study

$$\Theta^{00} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^0 \phi - \mathcal{L} = \frac{\partial \mathcal{L}}{\partial(\dot{\phi})} \dot{\phi} - \mathcal{L} = \mathcal{H} \tag{109}$$

the Hamiltonian for the field; in other words, the energy density of the field. We see that invariance to time translation, δa^0, implies energy conservation. The full conservation law must therefore be the conservation of energy-momentum for the field. We thus refer to $\Theta^{\mu\nu}$ as the *energy-momentum tensor.*

As a specific example, consider a real Klein-Gordon field with Lagrangian $\mathcal{L} = (1/2)[\partial^\mu\phi\partial_\mu\phi - m^2\phi^2]$. The energy-momentum tensor is then

$$\Theta^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} \tag{110}$$

In particular, we see that $\rho = \Theta^{00} = \mathcal{H} = (1/2)\dot{\phi}^2 + (1/2)(\nabla\phi)^2 + (1/2)m^2\phi^2$.

Now consider a complex KG field with $\mathcal{L} = \partial^\mu\phi^*\partial_\mu\phi - m^2|\phi|^2$. This Lagrangian, and hence the action, is invariant under a complex rotation $\phi \to \phi' = e^{i\alpha}\phi$. Since this is a continuous one-parameter invariance of the action, we expect there to be an associated conserved current, which we find as follows:

$$\delta\phi = i\alpha\phi \tag{111}$$

$$\delta\partial_\mu\phi = i[\phi\partial_\mu\alpha + \alpha\partial_\mu\phi] \tag{112}$$

Applying this to the action integral, we find

$$\delta S = 0 = \int d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi} i\alpha\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} i(\alpha\partial_\mu\phi + \phi\partial_\mu\alpha) \right\} \tag{113}$$

Quantization of Free Fields

In this section we give a small taste of second quantization, which is to say, field quantization. We use the canonical quantization technique based on the Hamiltonian, a method familiar from the development of quantum mechanics. We focus at first on a real, free scalar field satisfying the Klein-Gordon equation. The Lagrange density for this field is (see above)

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi - m\phi^2\right) \tag{114}$$

We define the conjugate momentum as

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi = \partial_0\phi \tag{115}$$

and construct the Hamiltonian density

$$\mathcal{H} = \pi\partial_0\phi - \mathcal{L} = \frac{1}{2}\left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right) \tag{116}$$

(recall the expression above for Θ^{00}). The Hamiltonian for the field is then

$$H = \int d^3x\mathcal{H} \tag{117}$$

As in quantum mechanics, these quantities become operators when we quantize the field. In other words, the field ϕ and its conjugate momentum become operators, as do all expression involving them, most notably, of course, the Hamiltonian. Quantization of the field is done by imposing the fundamental commutation relation between conjugate variables

$$[\phi(t,\vec{x}), \pi(t,\vec{y})] = i\delta(\vec{x} - \vec{y}) \tag{118}$$

This is called an *equal-time commutator,* because the two fields are taken on the same time hypersurface; we also have $[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\pi(\mathbf{x}), \pi(\mathbf{y})] = 0$, where these apply at arbitrary times. It is important to note that canonical quantization treats time and space separately in this way.

Recall that the general solution for the classica real scalar field of the KG equation has a general solution

$$\phi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a(\vec{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} + a^*(\vec{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \right) \tag{119}$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$ and we choose the *k-space measure* shown for convenience in the following. The quantum field should then be written

$$\phi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a(\vec{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\vec{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \right) \tag{120}$$

where a and $a^\dagger$ are now operators. The conjugate momentum is

$$\pi(\mathbf{x}) = -\int \frac{d^3k}{(2\pi)^3 2} i \left(a(\vec{k}) e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\vec{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \tag{121}$$

and the fundamental commutation relation is satisfied with

$$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3 2\omega_k \delta(\vec{k} - \vec{k}') \tag{122}$$

$$[a(\vec{k}), a(\vec{k}')] = [a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0 \tag{123}$$