Classical Field Theory

Paris Denis Diderot **2008**

Midterm exam homework

1. **Green Function for static solutions to the Klein-Gordon equation with source**
   The Klein-Gordon equation with a source ρ is $(\Box + m^2)\phi = \rho$, and static solutions satisfy $(-\nabla^2 + m^2)\phi(\vec{x}) = \rho(\vec{x})$.
   a. Demonstrate that a solution is given by $\phi(\vec{x}) = \int d^3x'\, G(\vec{x},\vec{x}')\rho(\vec{x}')$, where $G(\vec{x},\vec{x}')$ is the Green function: $(-\nabla^2 + m^2)G(\vec{x},\vec{x}') = \delta(\vec{x}-\vec{x}')$.
   b. Using the Fourier decomposition for the Green function
   $$G(\vec{x},\vec{x}') = \frac{1}{(2\pi)^{3/2}} \int d^3k\, G(\vec{k},\vec{x}')e^{i\vec{k}\cdot\vec{x}}$$
   show that
   $$G(\vec{k},\vec{x}') = \frac{1}{(2\pi)^{3/2}} \frac{e^{-i\vec{k}\cdot\vec{x}'}}{k^2+m^2}.$$
   c. Perform the integration over angles to find
   $$G(\vec{x},\vec{x}') = \frac{-i}{(2\pi)^2}\frac{1}{R}\frac{1}{2}\int_{-\infty}^{\infty} dk\, \frac{k}{k^2+m^2}\left(e^{ikR} - e^{-ikR}\right)$$
   where $R \equiv |\vec{x}-\vec{x}'|$.
   d. Identify the poles and residues of the remaining integration over dk. Perform the integral to find the final expression for the Green function.
   e. What is the potential, i.e., the value of the field ϕ, of a "point source" q placed at the origin? Explain in which sense $R_c \equiv 1/m$ is a characteristic scale of this potential. Comment the limit $m \to 0$.

2. **Covariant current of a charged particle**
   Consider a point-like particle of charge q moving along a trajectory $\vec{r}(t)$ in a given reference frame. Such a particle generates an electromagnetic field $A^\mu(\mathbf{x})$ at an arbitrary space-time point $\mathbf{x} = (x^0, \vec{x})$.
   a. Write the charge density $\rho(\mathbf{x})$ and the current density $\vec{j}(\mathbf{x})$ in terms of the trajectory $\vec{r}(t)$ and Dirac delta functions.
   b. The particle coordinates can be written as a 4-vector $r^\mu(\tau) = [r^0(\tau), \vec{r}(\tau)]$ where τ is the proper time of the particle. Show that the 4-current is then given by
   $$J^\mu(\mathbf{x}) = q\int d\tau\, U^\mu(\tau)\delta^{(4)}[\mathbf{x} - \mathbf{r}(\tau)]$$
   c. Verify that this is indeed a 4-vector. Note that the transformation law for a 4-vector (as usual, but more explicitly) is $J^\mu(\mathbf{x}') = \Lambda^\mu{}_\nu J^\nu(\mathbf{x})$, where the coordinates, of course, are related by the Lorentz Transformation, $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

3. Electromagnetic Lagrangian

The Lagrangian for a free electromagnetic field (no sources) is

$$\Im = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where, as usual, $F_{\mu\nu} = \partial_\mu \mathrm{A}_\nu - \partial_\nu A_\mu$ is the Maxwell tensor.

a. Evaluate $\partial(\partial_\mu \mathrm{A}_\nu) / \partial(\partial_\alpha A_\beta)$.

b. Show that $\dfrac{\partial \Im}{\partial(\partial_\alpha A_\beta)} = -F^{\alpha\beta}$.

c. Find the field equations for the 4-potential A_μ. What is the result in the Lorentz gauge: $\partial_\mu A^\mu = 0$?

d. Find the term that must be added to the Lagrangian to yield the field equation (i.e., the Lagrange equations) for the 4-potential with source J^μ. Prove your result.