NPA_Cosmologie : *25 février 2008* Année 2008

Structures et leur évolution

Cours 6 : Evolution des perturbations au régime relativiste

Why/When Do We Need It?

- Strong gravitational fields: $|\Phi| \geq 1$
  - *Not our case*
- High velocities: $|v| \to 1$
  - *Not our case, except…*
  - High pressure fluid (high thermal velocities)
    - **Baryon-photon fluid at recombination** $p \sim p_\gamma \sim \rho_\gamma \sim \rho_B$
- "Acausal" perturbations
  - **Wavelengths larger than the horizon** $a\lambda \geq H^{-1} \sim t$
- Other gravitational modes: *e.g., tensors=grav. waves*

General Relativity:

Our notation

$\boxed{c = 1}$

Metric signature: $\eta = diag(1,-1,-1,-1)$

Affine connection: $\Gamma^{\delta}_{\mu\nu} = \frac{1}{2} g^{\delta\alpha}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - 2g_{\mu\nu,\alpha}\right)$

Riemann curvature: $R^{\mu}_{\alpha\beta\gamma} = \Gamma^{\mu}_{\alpha\gamma,\beta} - \Gamma^{\mu}_{\alpha\beta,\gamma} + \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\gamma\alpha} - \Gamma^{\mu}_{\sigma\gamma}\Gamma^{\sigma}_{\beta\alpha}$

Einstein field equations: $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$

$\Rightarrow T^{\mu\nu}{}_{;\nu} = 0$ energy conservation

$; =$ covariant derivative

$, = \partial_i$

Basics of the Friedmann Model

The C.P. applied in its strictest sense implies that *space* is homogeneous and isotropic (we will add perturbations later). If we describe the cosmic contents as a perfect fluid, then there must be a reference frame in which

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

$$\rho = \rho\left(x^0\right) \;\; \& \;\; p\left(x^0\right)$$

I.e., constant on spatial hypersurfaces.

Covariant expression: $T^{\mu\nu} = (\rho + p) U^\mu U^\nu - p g^{\mu\nu}$ $\quad U^\mu = \text{ fluid 4-velocity}$

Spacetime Metric

C.P. $\Rightarrow$ $\ni$ a system of homogeneous/isotropic coordinates: (r, θ, φ)

(this is the system in which the fluid is at rest: $U^i = 0$)

Comoving coordinates

$$ds^2 = dt^2 - a^2(t)dl^2$$

$$= dt^2 - a^2(t)\left[\frac{dr^2}{1-\mathrm{K}r^2} + r^2 \overbrace{\left(d\theta^2 + \sin^2\theta d\varphi^2\right)}^{\text{Isotropic}}\right]$$

Scale factor

Spatial curvature (1/distance²)

$\mathrm{K} = 1/R_{curv}^2 = const.$ Homogen.

$\mathrm{K} < 0:$	Hyperbolic space (open)	$r < \infty$	*Infinite space*
$\mathrm{K} = 0:$	Flat space	$r < \infty$	
$\mathrm{K} > 0:$	Spherical space (closed)	$r < R_{curv}$	*Finite space*

Another form:

$$ds^2 = dt^2 - a^2(t)\left[d\chi^2 + S^2(\chi)\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right]$$

$$S(\chi) = \sinh(\chi) \quad \mathrm{K}<0$$
$$S(\chi) = \chi \quad \mathrm{K}=0$$
$$S(\chi) = \sin(\chi) \quad \mathrm{K}>0$$

The expansion:

$$\boxed{dl = a(t)d\chi}$$

The proper separation between two fixed coordinates increases

The redshift:

$$1+z \equiv \frac{a_0}{a(t)} \qquad a_0 \equiv a(today)$$

Conformal time:

$$d\tau \equiv dt\ /\ a(t)$$
$$ds^2 = a^2(t)\left(d\tau^2 - dl^2\right)$$

Evolution: Friedmann equations

The Einstein field equations:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\mathrm{K}}{a^2} + \frac{\Lambda}{3} \equiv H_0^2 E^2(a)$$

Hubble parameter

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0$$

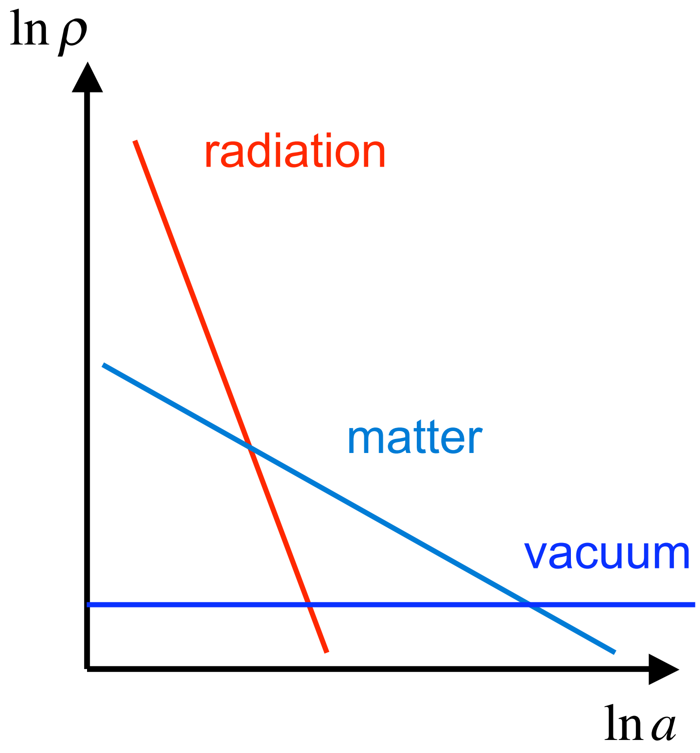
Energy conservation equation

Three unknown functions, *a(t), ρ(t), p(t)*, but only 2 independent equations; system is closed by the equation-of-state *p[ρ].*

Evolution: Solutions

Equation-of-state (simple & useful example): $p = w\rho \quad w = const.$

$$\rho = \rho_0 a^{-3(1+w)}$$



Non-relativistic matter ("dust"): $p = 0$

$$\boxed{\rho = \rho_0 a^{-3} = \rho_0 (1+z)^3} \qquad \boxed{a(t) \propto t^{2/3}}$$

Radiation: $p = \frac{1}{3}\rho$

$$\boxed{\rho = \rho_0 a^{-4} = \rho_0 (1+z)^4} \qquad \boxed{a(t) \propto t^{1/3}}$$

Vacuum energy: $p = -\rho$

$$\boxed{\rho, H = const} \qquad \boxed{a(t) \propto e^{Ht}}$$

Thermal History

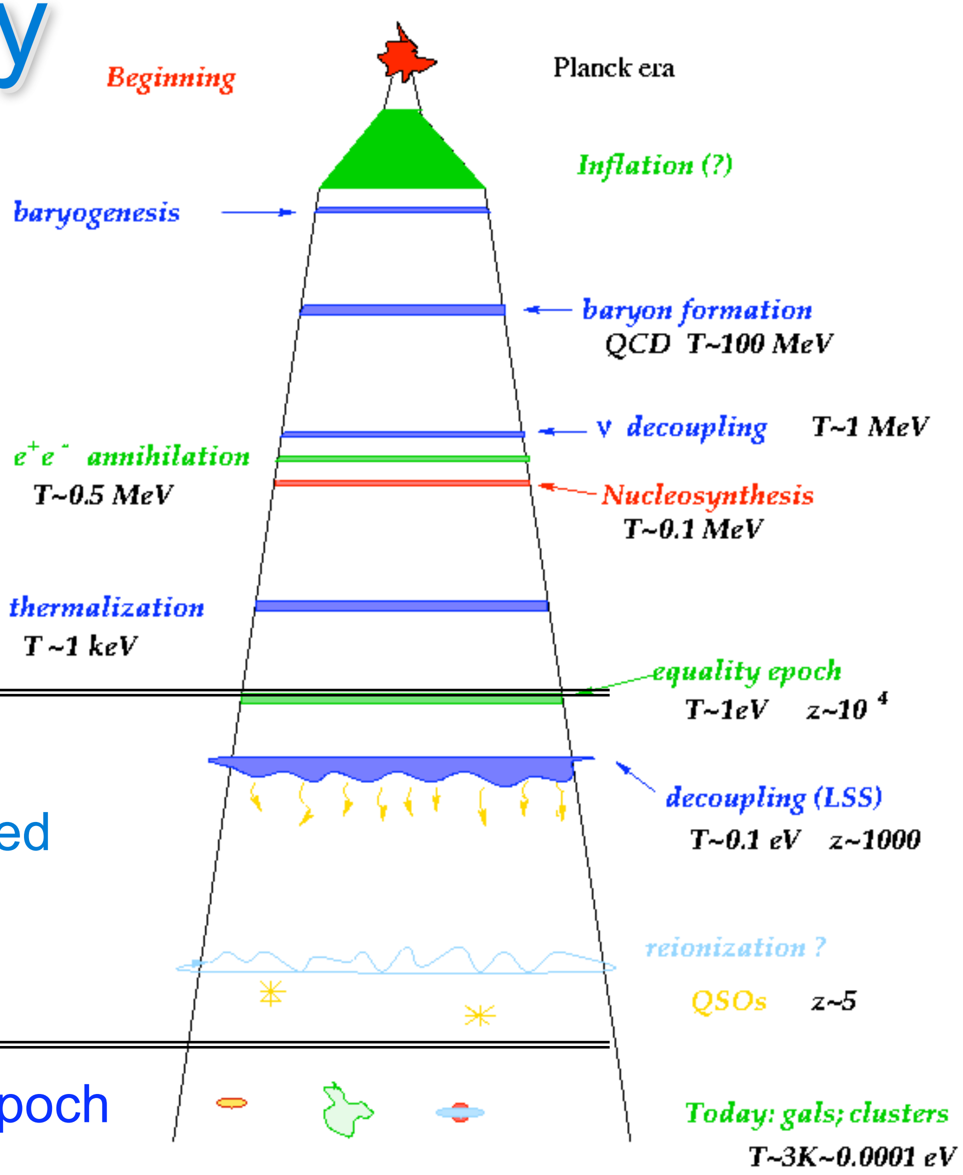
Radiation dominated epoch (RD)

$z \sim few \times 1000$

Matter dominated epoch (MD)

$z \sim 1$

Vacuum dominated epoch (dark energy)



The Cosmological Parameters

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3} = H_0^2\left[\Omega_R(1+z)^4 + \Omega_M(1+z)^3 + \Omega_K(1+z)^2 + \Omega_\Lambda\right] = H_0^2 E^2(z)$$

The density parameters:

$$\Omega_M \equiv \frac{8\pi G\rho_{M,0}}{H_0^2} \qquad \Omega_\Lambda \equiv \frac{\Lambda}{3H_0^2} \qquad \Omega_K \equiv -\frac{K}{H_0^2 a_0^2} \qquad \Omega_R \equiv \frac{8\pi G\rho_{R,0}}{H_0^2} \quad (\ll \Omega_M, \Omega_\Lambda)$$

Note that physical density $\rho_I \propto \Omega_I H_0^2$

$$\Omega_M = \Omega_X + \Omega_B$$

Constraint: $1 = \Omega_M + \Omega_\Lambda + \Omega_K = \Omega_{TOT} + \Omega_K$

The Hubble parameter: $H_0 \equiv h \times 100\, km / s / Mpc$

$$\boxed{\Omega_M, \Omega_B, \Omega_\Lambda, H_0}$$

The Standard Model

Inflationary epoch

- Flatness problem
- Horizon problem
- Monopole problem
- Perturbation generation

Diverse observations

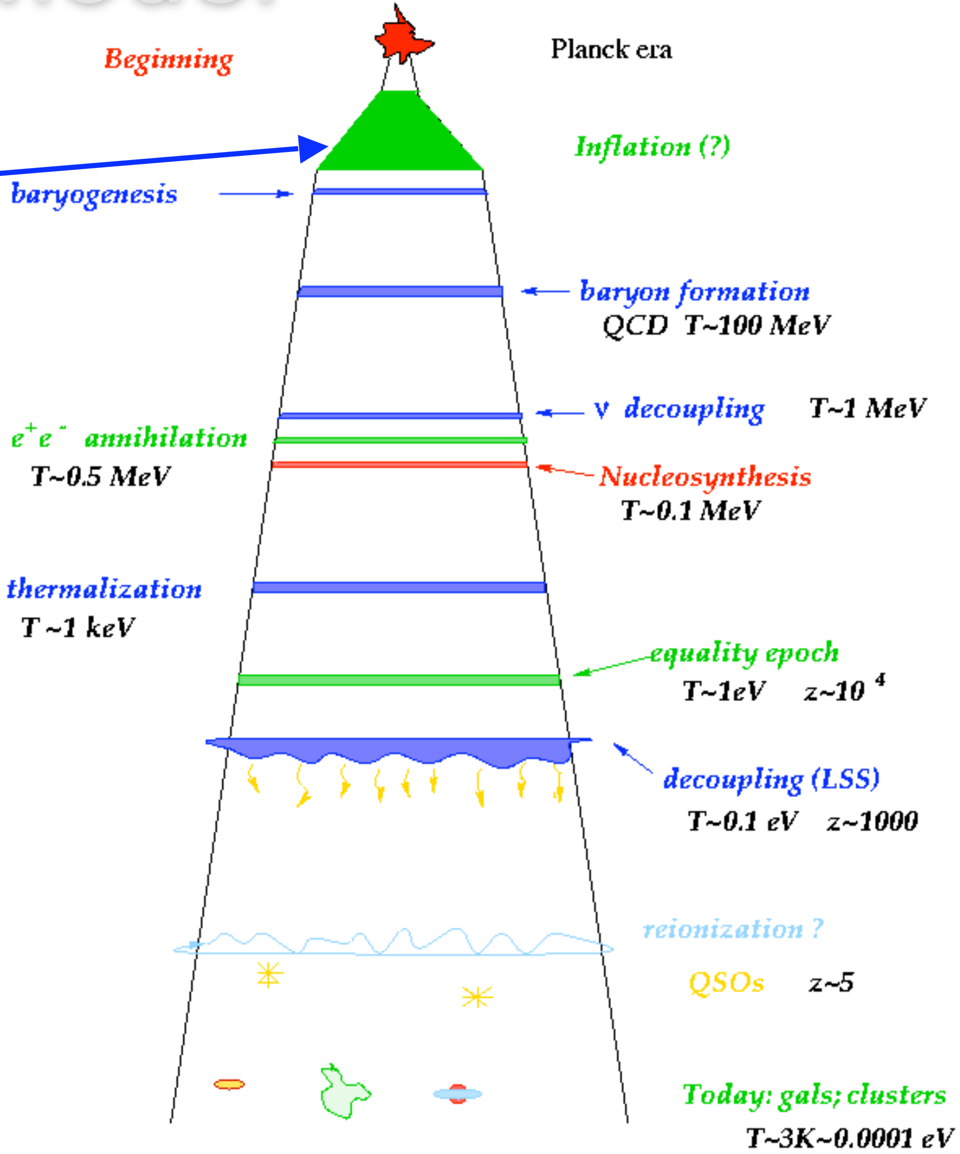
$$\Omega_M \simeq 0.3$$

$$\Omega_\Lambda \simeq 1 - \Omega_M \simeq 0.7$$

$$\Omega_K \simeq 0 \qquad \text{i.e., Flat space}$$

$$h \simeq 0.7$$

- Over-constrain the model
- Our job ahead



Density Perturbations & Their Evolution

- *Primordial* spectrum: $P_i(k)$
  - Inflation physics

- Transfer function *T(k)*
  - Gravitational evolution ("passive")

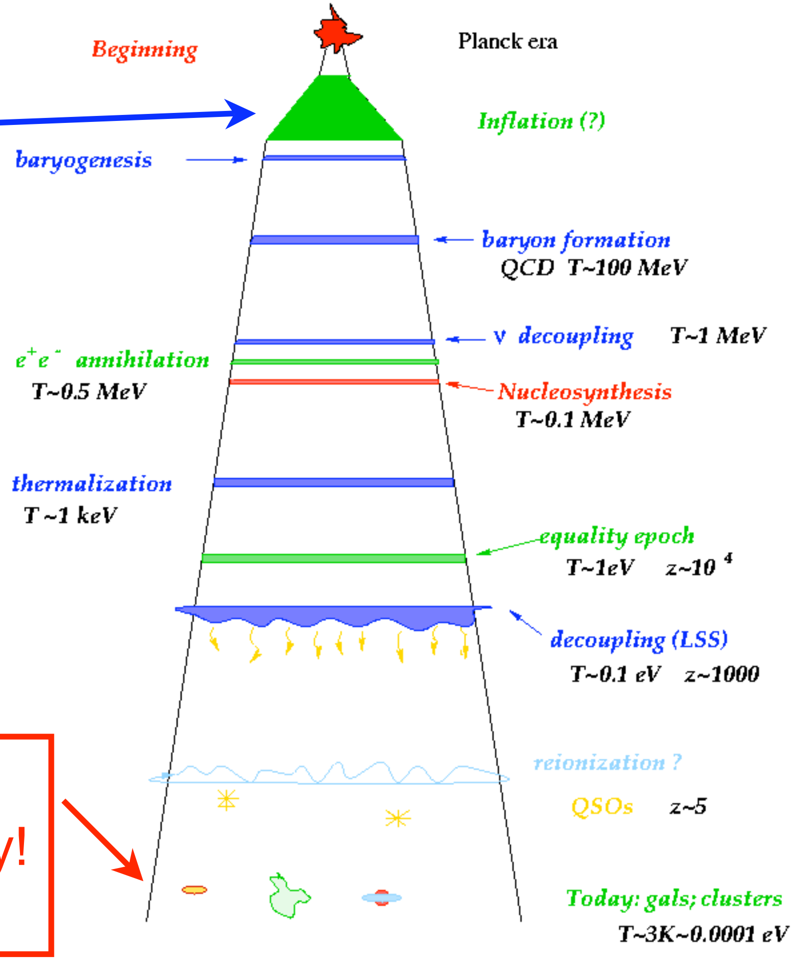
- *Final* spectrum:

$$P(k) = T^2(k) P_i(k)$$

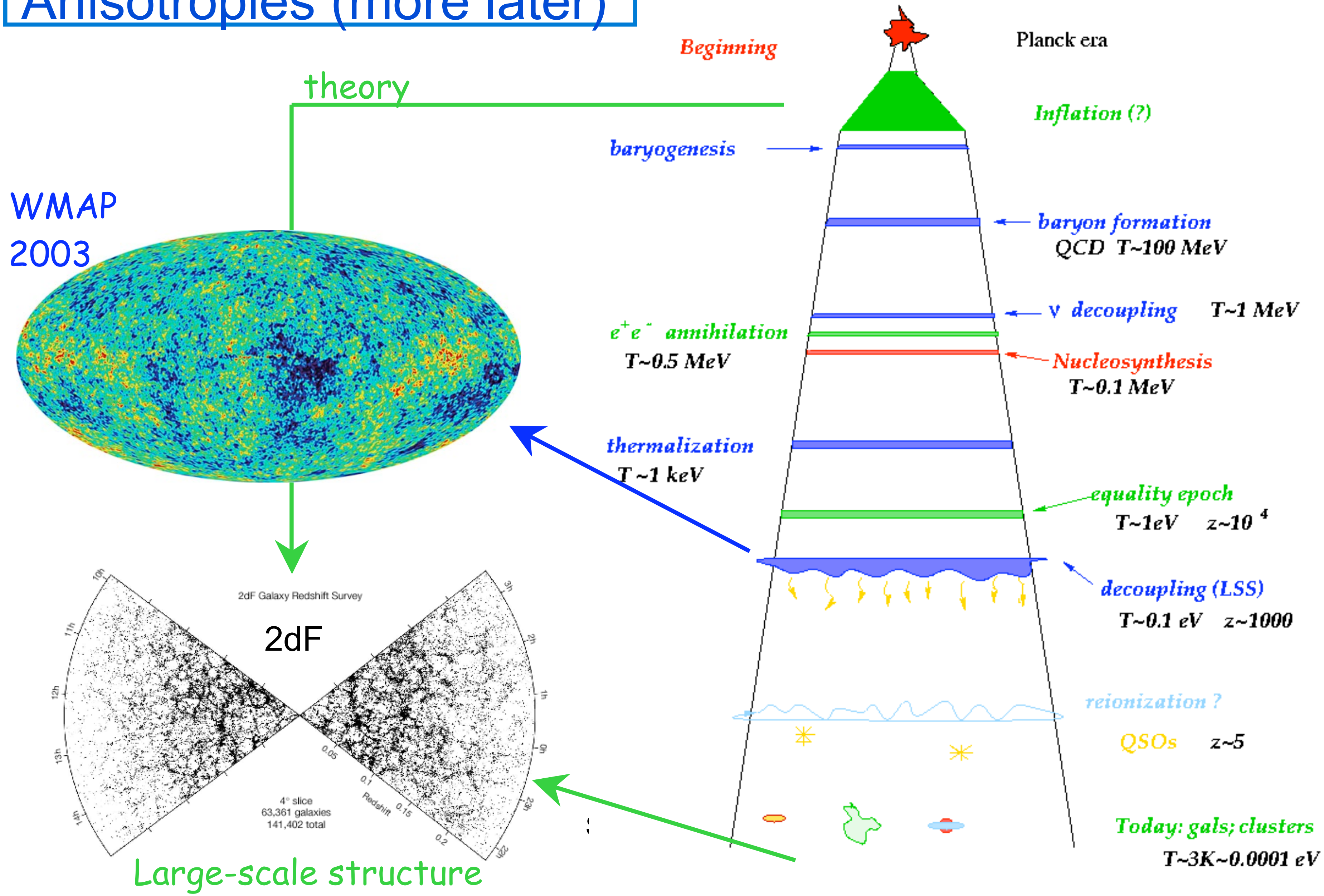
Observations

- *Success* of theory!

- Cosmo params.



Relation to CMB Anisotropies (more later)



theory

WMAP
2003

2dF

Large-scale structure

Beginning

Planck era

Inflation (?)

baryogenesis

baryon formation
QCD T~100 MeV

ν decoupling T~1 MeV

e^+e^- annihilation
T~0.5 MeV

Nucleosynthesis
T~0.1 MeV

thermalization
T ~1 keV

equality epoch
T~1eV z~10^4

decoupling (LSS)
T~0.1 eV z~1000

reionization ?

QSOs z~5

Today: gals; clusters
T~3K~0.0001 eV

Inflation

<u>In its basic conception</u>: universe is dominated by a uniform scalar field φ

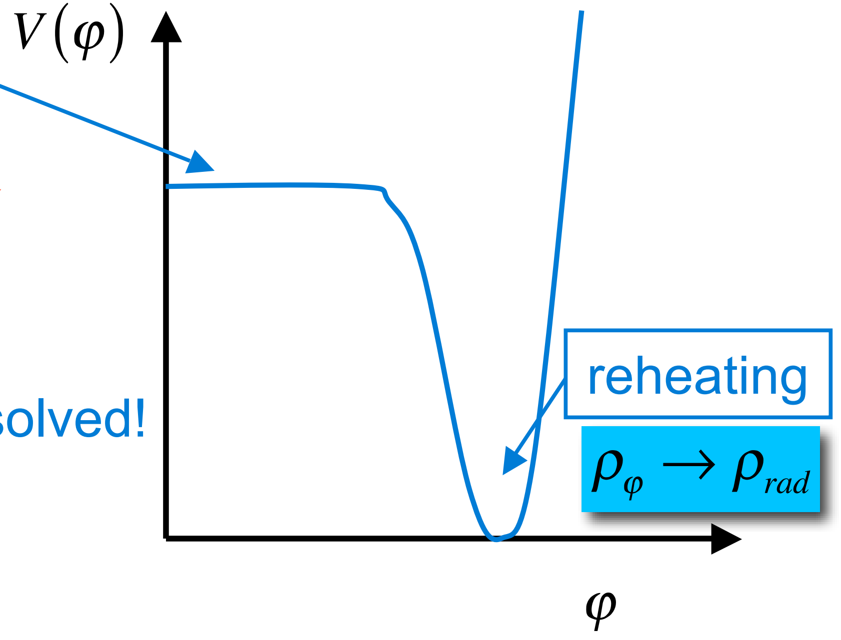
$$\boxed{\rho_\varphi = V(\varphi) \simeq -p \simeq const}$$

"Slow roll"

$$H^2 = \frac{8\pi G}{3}\rho_\varphi \simeq const$$

$$\boxed{a(t) \propto e^{Ht}}$$

Exponential expansion



$$d_H(t) = a(t)\int_0^t \frac{dt'}{a(t')} \sim H^{-1}e^{Ht}$$

1. Horizon solved!

$$\frac{K/a^2}{\rho_\varphi} \to 0$$

2. Flatness solved!

$$\frac{\rho_{monopoles}}{\rho_\varphi} \propto a^{-3} \to 0$$

3. Monopoles solved!

4. Perturbations

Perturbation Theory: *the metric*

$$ds^2 = a^2(\tau)\left[d\tau^2 - \gamma_{ij}dx^i dx^j\right]$$

Unperturbed metric in comoving coordinates

γ_{ij} - Spatial metric; *we will take K=0 - ok at early times*

In Cartesian coordinates: $\gamma_{ij} = \delta_{ij}$

τ - Conformal time

General perturbed metric

$$ds^2 = a^2(\tau)\left\{(1+2\Psi)d\tau^2 - \left[(1-2\Phi)\gamma_{ij} + 2h_{ij}\right]dx^i dx^j - \omega_i d\tau dx^i\right\}$$

$\{\Psi,\ \Phi,\ \omega_i,\ h_{ij}\}(\tau, x^i)$ Are *small* perturbations defined on the *background 3-space: e.g.,* $h^i{}_j = \gamma^{ij}h_{ij}$

First order pert. theory

The perturbation functions $\{\Psi,\ \Phi,\ \omega_i,\ h_{ij}\}(\tau, x^i)$

To first order:

- All indices are raised/lowered with: γ_{ij}
- Covariant derivative in 3-space = ∂_i (Flat Cartesian coordinates)

Counting degrees of freedom:

Φ, Ψ 2 functions

ω_i 3 functions

h_{ij} $9 - \underbrace{\textcircled{3}}_{\text{symmetric}} - \underbrace{\textcircled{1}}_{Tr(h)=0} = 5$ functions

10 degrees of freedom

Decomposition:

$$\omega_i = \partial_i a + b_i \quad \text{with} \quad b_i^{;i} = 0$$

$$h_{ij} = \left(\partial_i \partial_j - \frac{1}{3}\gamma_{ij}\nabla^2\right)h + \left(h_{i;j} + h_{j;i}\right) + h_{ij}^T$$

with $h_i^{;i} = 0$ $h_{ij}^{T;j} = 0$

Ψ, Φ, a, h	Scalar perturbations
b_i, h_i	Vector perturbations
h_{ij}^T	Tensor perturbations

Gauge freedom: choice of coordinates in perturbed space

- Sychronous gauge: $\Psi = 0, \;\; \omega_i = 0$
- Poisson gauge: $\omega_i^{\;;i} = 0, \;\; h_{ij}^{\;\;;j} = 0$
- Longitudinal gauge: $a = 0, \;\; h = 0$ (Scalar perturbations only)

Can work in a particular gauge, or with gauge invariant quantities

In standard model: only scalar (inflaton) & tensor (grav. wave) perturbations

We will adopt a gauge with: $a = 0, \; b_i = 0, \; h = 0$
(4 conditions)

$$ds^2 = a^2(\tau)\left\{(1+2\Phi)d\tau^2 - \left[(1-2\Phi)\gamma_{ij} + 2h^T{}_{ij}\right]dx^i dx^j\right\}$$

Newtonian-like potential

Grav. waves

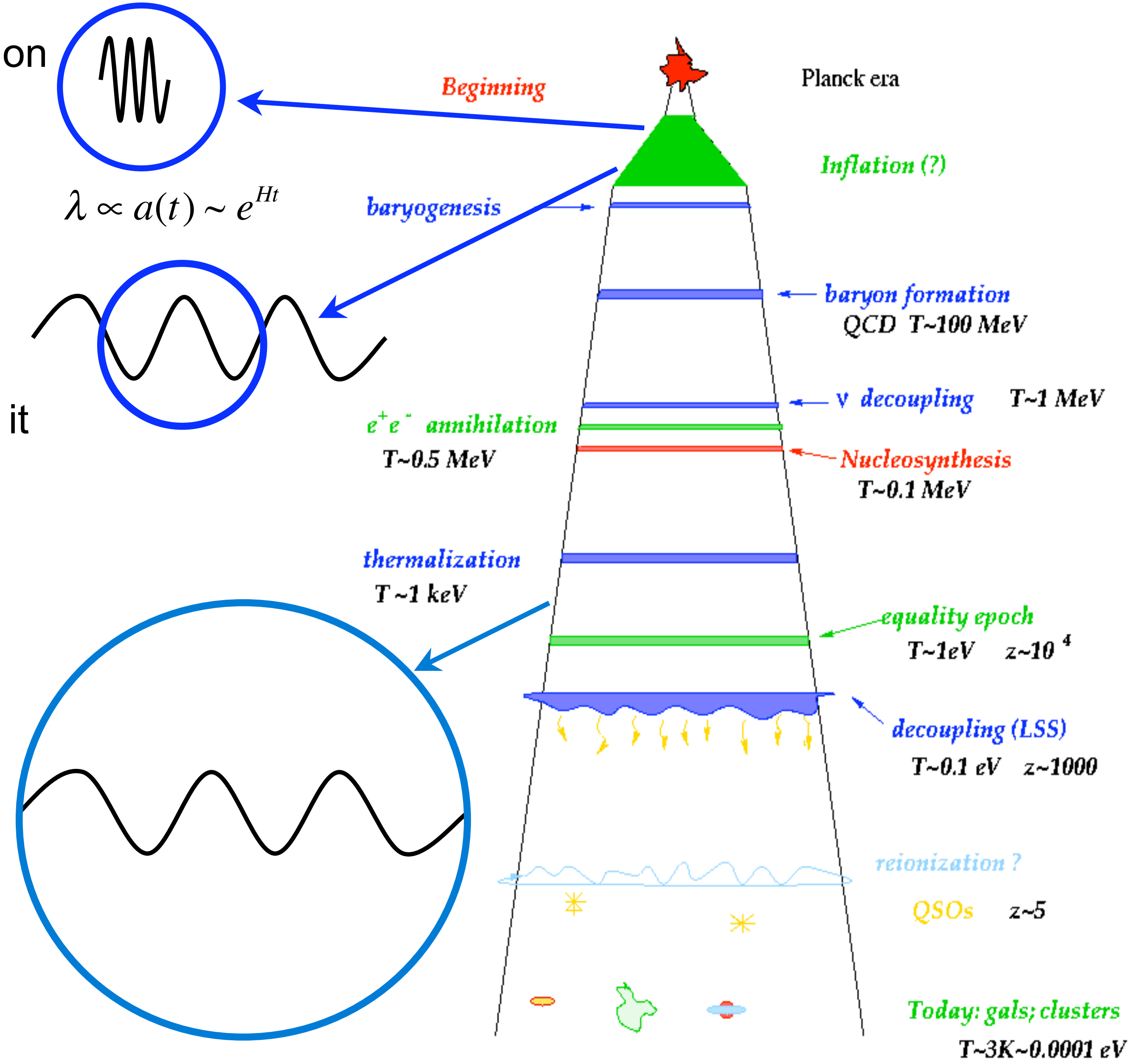
h_i is not sourced; $\Psi = \Phi$ in absence of anisotropic stress

1. Quantum fluctuation
inside horizon
(Causal physics)

Horizon size fixed during inflation

$$d_H \sim H^{-1} = c^{ste}$$

2. Expansion drives it
outside horizon
(No causal physics)



$$\lambda \propto a(t) \sim e^{Ht}$$

3. Horizon grows faster than λ
reenters horizon
(Causal physics)

$$d_H \sim H^{-1} = t$$

$$\lambda \propto a(t) \sim t^n \quad n < 1$$

Perturbation Theory: *the matter*

Perfect fluid approximation: $T^{\alpha\beta} = (\rho + p)U^{\alpha}U^{\beta} - pg^{\alpha\beta}$

➡ $$T^{\alpha}{}_{\beta} = \begin{pmatrix} \rho_0 + \delta\rho & (\rho_0 + p_0)a^{-1}v^i \\ (\rho_0 + p_0)av_i & -(p_0 + \delta p)\delta^i_j \end{pmatrix}$$

$$p(\rho, S) \quad \text{equation of state} \quad \Rightarrow \delta p = \boxed{c_s^2\delta\rho} + \boxed{\eta\delta S}$$

entropy pert. (e.g., changes in the particle ratios)

adiabatic pert.

$T^{\alpha}_{\beta\,;\alpha} = 0$ ➡ For each separate component of the fluid:

$$\delta' + 3\frac{a'}{a}\left(c_s^2 - x\right)\delta + (1 + x)\left(\nabla \cdot \vec{v} - 3\Phi'\right) = -3\frac{a'}{a}\frac{1}{\rho_0}\eta\delta S$$

$$\frac{1}{a^4}\partial_\tau\left[a^4\rho_0(1 + x)\vec{v}\right] + c_s^2\rho_0\nabla\delta + \rho_0(1 + x)\nabla\Phi = -\eta\nabla\delta S$$

Einstein Equations

$$h_{ij}'' - \nabla^2 h_{ij} + 2\frac{a'}{a}h_{ij}' = 0$$

Tensor = gravity waves (more later)

$$\nabla^2\Phi - 3\frac{a'}{a}\left(\Phi' + \frac{a'}{a}\Phi\right) = 4\pi G a^2 \rho_0 \delta$$

Scalar perturbations

Adiabatic perturbations outside the horizon $\left(k < \tau^{-1} \sim aH\right)$:

$$\left.\begin{aligned} &\Phi' + \frac{a'}{a}\Phi = -\frac{1}{2}\frac{a'}{a}\delta \\ &\delta' + 3\frac{a'}{a}\left(c_s^2 - x\right)\delta - 3(1+x)\Phi' = 0 \end{aligned}\right\} \quad \Phi = -\frac{1}{2}\delta = cste$$

$$\frac{1}{a^4}\partial_\tau\left[a^4\rho_0\left(1+x\right)\vec{v}\right] = 0$$

Perturbations reenter with the same amplitude they had upon exit

Inflationary Perturbations

Inflation's predictions:

- Adiabatic perturbations: $\delta p(\rho, S) = c_s^2 \delta\rho + \eta \delta S = c_s^2 \delta\rho$
  Scalar field dominates total energy density
- Gaussian statistics: $P\left(\delta_{\vec{k}_1}, \delta_{\vec{k}_2}, \ldots\right) \propto \prod \exp\left[-\frac{1}{2}\frac{\delta_{\vec{k}_i}{}^2}{P(k_i)}\right]$

  Ground (vacuum) state of (almost) free scalar field
- Scale invariant: $\Delta_\Phi^2\left(k = a_r H(t_r)\right) \sim cste$ @ reentry
  Vacuum perturbations & near time-translation invariance of inflation

At horizon exit

$$\Delta^2\Phi \equiv \frac{k^3\left|\Phi\left(\vec{k}\right)\right|^2}{2\pi^2} \sim \frac{H^4}{2\pi\dot{\varphi}^2}\bigg|_{k\sim H(t_e)a_e}$$

After reheating

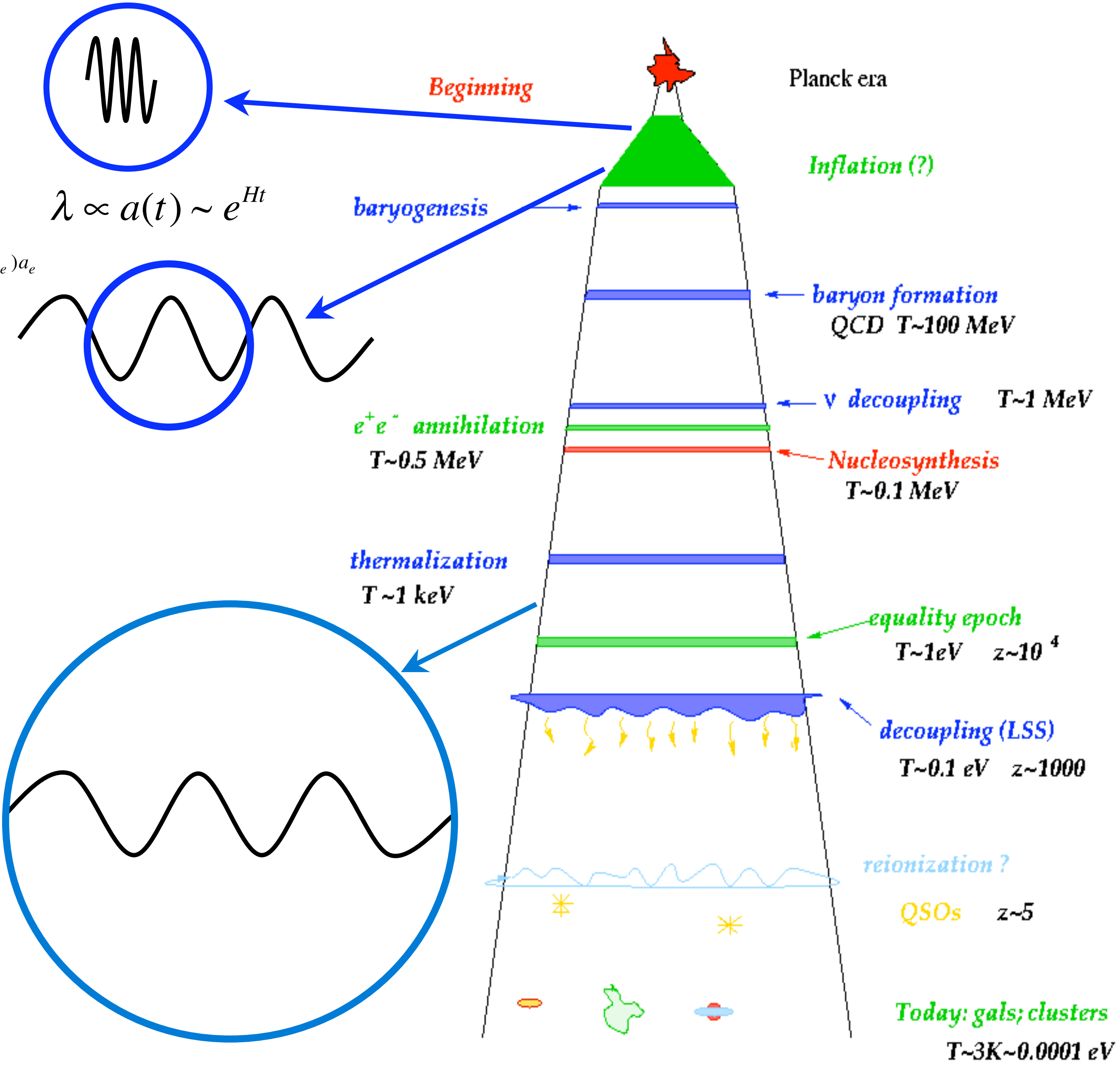
$$\Phi = cste$$

At horizon rentry

$$\Delta^2\Phi\big|_{k\sim H(t_r)a_r} \sim A\frac{H^4}{2\pi\dot{\varphi}^2}\bigg|_{k\sim H(t_e)a_e}$$

$C^{ste}(k)$

$$\lambda \propto a(t) \sim t^n \qquad n < 1$$



$$\lambda \propto a(t) \sim e^{Ht}$$

Subhorizon Evolution $\left(k > \tau^{-1} \sim aH\right)$

Before recombination ($z>z_{rec}$):

Baryon-photon fluid: *photon pressure drives acoustic wave oscillations*

Dark matter: $\left(c_s^2 = x = 0\right)$

$$\left.\begin{aligned}
&\delta'_{DM} + \left(\nabla \cdot \vec{v}_{DM} - \cancel{3\Phi'}\right) = 0 \\
&\frac{1}{a^4}\partial_\tau\left[a^4 \bar{\rho}_{DM} \vec{v}_{DM}\right] + \bar{\rho}_{DM} \nabla\Phi = 0 \\
&\nabla^2\Phi - 3\frac{a'}{a}\left(\cancel{\Phi' + \frac{a'}{a}\Phi}\right) = 4\pi G a^2 \rho_0 \delta
\end{aligned}\right\}$$

(Newtonian result!…)

$$\delta''_{DM} + \frac{a'}{a}\delta'_{DM} = \nabla^2\Phi = 4\pi G a^2 \rho_0 \sum_{\substack{i=DM,\\ B-\delta,\nu}} \Omega_i(\tau)\delta_i$$

$$\text{At } z \gg z_{eq}: \quad \delta_{\mathrm{DM}} \sim \ln(\tau) + cste$$

$$\text{At } z \ll z_{eq}: \quad \delta_{\mathrm{DM}} \sim \underset{\sim a(\tau)}{\boxed{\tau^2}},\ \tau^{-3}$$

After recombination ($z_{rec}>z>few$):

$\delta_{\mathrm{DM}}, \delta_B \sim a$ *& photons free stream*

Subhorizon Evolution

Radiation epoch

$\delta_{B-\gamma}(\vec{k})$ oscillates

$\delta_{DM}(\vec{k})$ grows logarithmically

Matter epoch

Before recombination

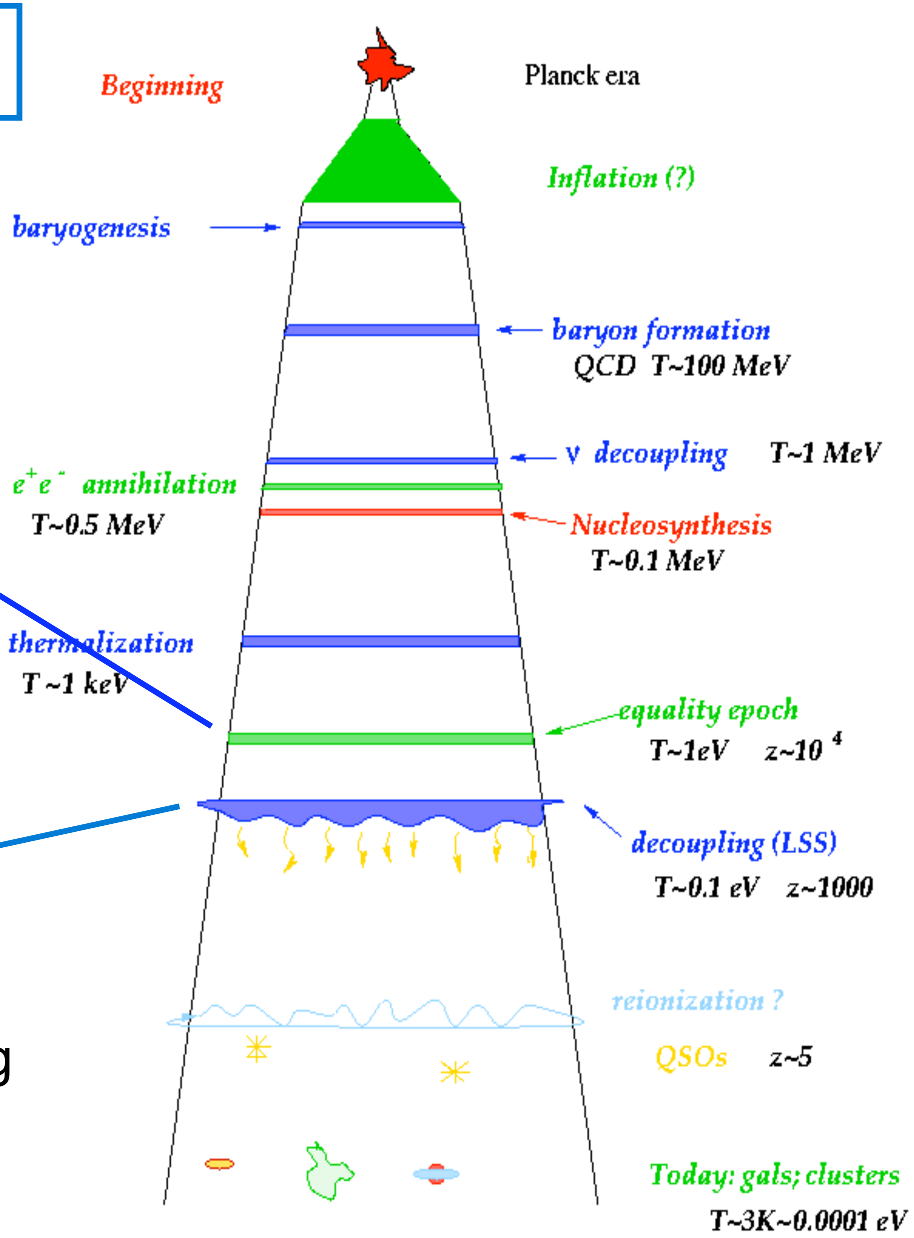
$\delta_{B-\gamma}(\vec{k})$ oscillates

$\delta_{DM}(\vec{k}) \propto a(t)$

After recombination (*z>few*)

$\delta_{\gamma}(\vec{k})$ decays by free-streaming

$\delta_{DM}(\vec{k}), \delta_B \propto a(t)$



Harrison-Zel'dovich (Peebles) Spectrum:

Long before the theory of inflation…

Simplicity (absence of motivated scale): $P(k) \propto k^{n_s}$ for $aH < k < k_{eq}$

HZ spectrum: $n_s = 1$ *scale invariant spectrum*

For $a>a_{eq}$:

$$\left.\begin{array}{l} \Delta^2(k,a) = \left(\frac{a}{a_r}\right)^2 \Delta^2(k,a_r) \\ k = a_r H(t_r) \sim a_r^{-1/2} \end{array}\right\} \Rightarrow$$

At reentry:

$$\boxed{\Delta^2(k,a_r) \propto k^{n_s - 1}}$$

$\therefore n_s = 1 \Rightarrow cste$ at reentry

and/or

In terms of potential

$$k^2 \Phi(\vec{k}) \sim \delta(\vec{k}) \Rightarrow \Delta^2_\Phi(k) \equiv \frac{k^3 P_\Phi(k)}{2\pi^2} \propto k^{n_s - 1}$$

$\therefore n_s = 1 \Rightarrow \Phi$ has a scale invariant spectrum, inside and outside horizon (and is *cste* in time in mat. era)

Inflation provides a *physical mechanism* that realizes a near HZ spectrum (*n_s~1).*

Formation of late-time CDM-like spectrum

Passive: Initial conditions, followed by pure grav. evolution

- Near scale-invariant (e.g., Inflation)
- Adiabatic: $n_i / n_j = const$
- Gaussian

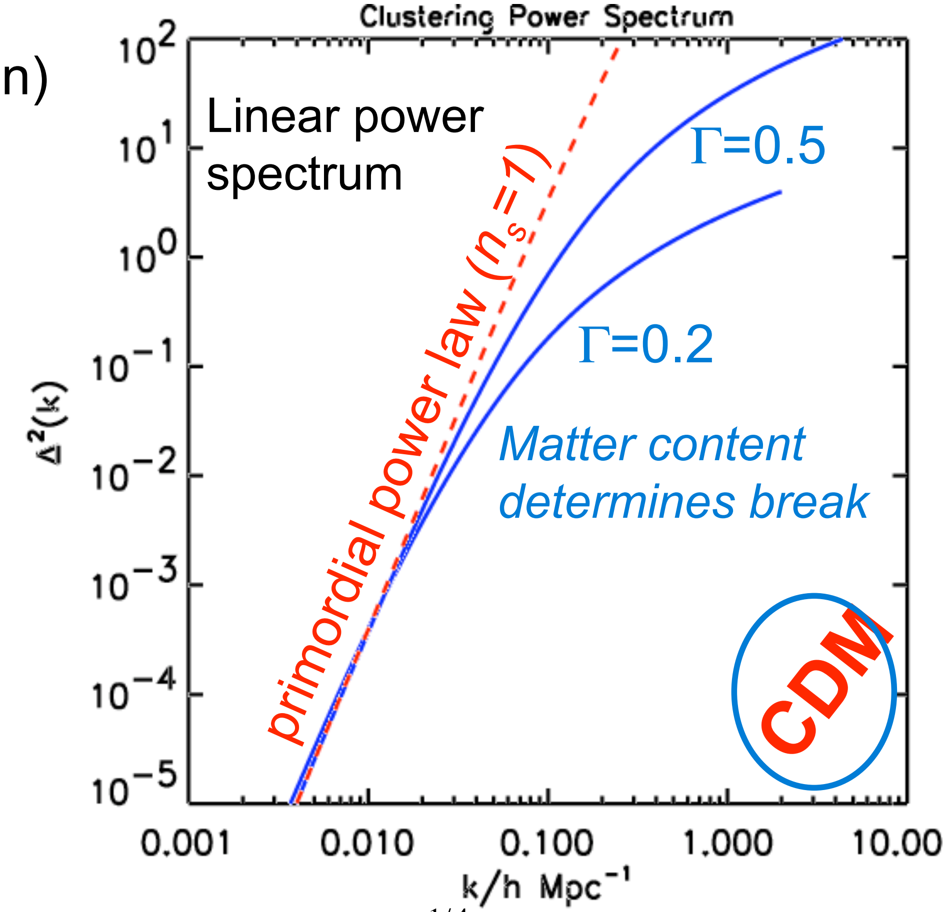
Late-time ($z > z_{eq}$) power spectrum for CDM (p=0)

$$P(k, z < z_{eq}) = T^2(k) A k^{n_s - 1}$$

T(k) = transfer function

$$q \equiv \frac{k}{\Omega h^2 \ \mathrm{Mpc}^{-1}} \equiv \frac{k}{(h \ \mathrm{Mpc}^{-1})} \frac{1}{\Gamma}$$

Γ = *shape parameter*

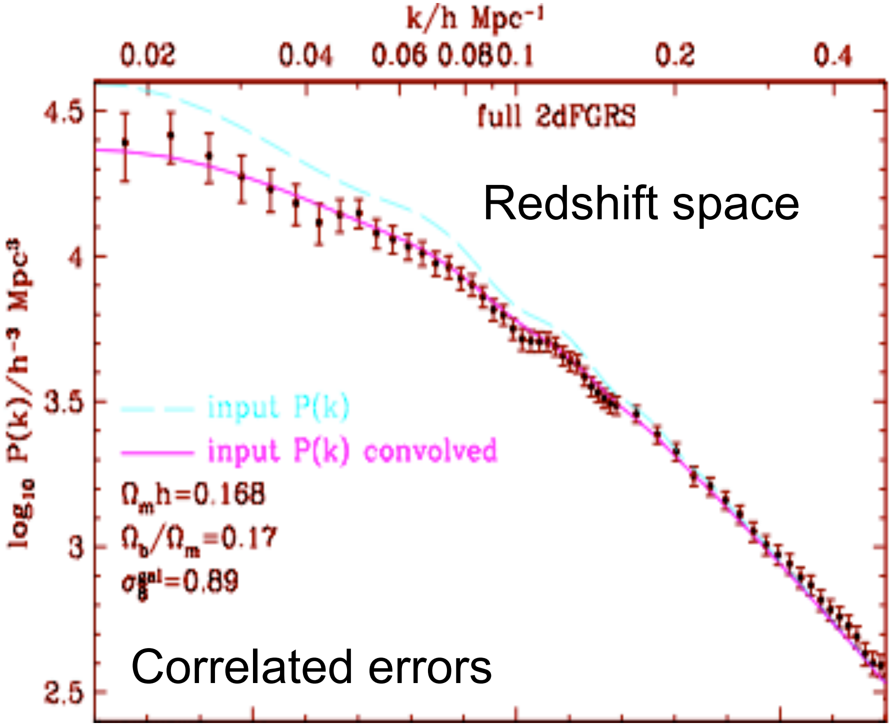


Matter content determines break

$$T(k) = \frac{\ln(1+2.34q)}{2.34q}\left[1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4\right]^{-1/4}$$ (BBKS)

Measured *Galaxy* Power Spectrum

2dF
221,414 galaxies

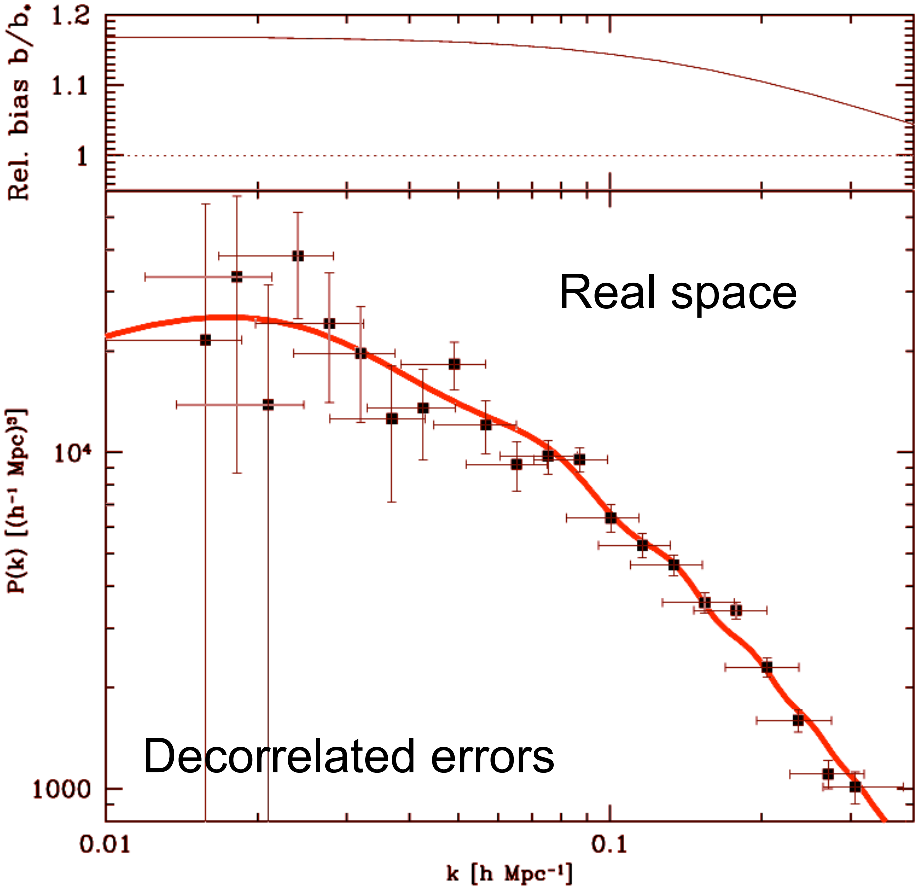


Redshift space

Correlated errors

Cole et al. 2005, MNRAS 362, 505

SDSS
200,000 galaxies (DR1)



Real space

Decorrelated errors

Tegmark et al. 2004, ApJ 606, 702