Chapter 3

Acceleration of cosmic-rays

3.1 Introduction

As we saw in the previous chapter, the cosmic-ray spectrum is one of the wonders and also one of the greatest mystery of modern physics. It extents over 12 decades in energy and 32 decades in flux, showing an almost perfect power law behavior only broken significantly at $\sim 3-4\times 10^{15}$ eV (the called *knee*) and $\sim 3-4\times 10^{18}$ eV (the so called *ankle*). Moreover cosmic-rays with energy of the order of $\sim 10^{20}$ eV ($\sim$ 16 J !) have been detected since the early 60's. With its characteristic shape and the extraordinary energies that can be reached in some cases, the cosmic-ray spectrum cannot be explained with thermal phenomena, which means that cosmic-rays we observe must have been accelerated in some way.

The most likely hypothesis is that this acceleration is due to electromagnetic fields present in astrophysical sources or in the interstellar medium. In the interstellar medium however the mean electric field $\langle E\rangle = 0$ since the interstellar ionized gas is almost perfectly conductor and globally neutral. Transient electric fields can be found for instance in solar flares (due to very complex magnetic reconnection phenomena) or as a result of locally varying magnetic fields. Finally, strong and lost lasting electric field are mostly found in the vicinity of neutron stars.

On the other hand, magnetic fields are found in all high-energy astrophysics sources as well as in the interstellar medium and are generally invoked in the most popular theoretical scenarios for cosmic-ray acceleration. This statement might look puzzling at first sight. Since the Lorentz force $\vec{F} = q\,\vec{v}\times\vec{B}$ does no work and then can in principle not be invoked to accelerate particles. However, as recalled above a time varying magnetic field induces an electric field as formalized in Maxwell's equation $\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$. Moreover, a pure magnetic field $\vec{B}'$ in a given reference frame is seen as a magnetic field $\vec{B}$ and an electric field $\vec{E}$ in another reference frame moving relative to it, as implied in electrodynamics by the Lorentz transformation of the electromagnetic tensor $F^{\mu\nu}$.

It is then important to understand that, although the calculations we will present in the following are mostly based on the description of the electromagnetic field in a given astrophysical medium in terms of a pure magnetic field, it is merely for technical convenience ($i.e$ because it makes the calculations easier). It is, in principle always possible to describe the same process in terms of electric fields which seems a priori more intuitively natural to explain charged particle acceleration. A simple example of the equivalence of the two approaches will be given in the following.

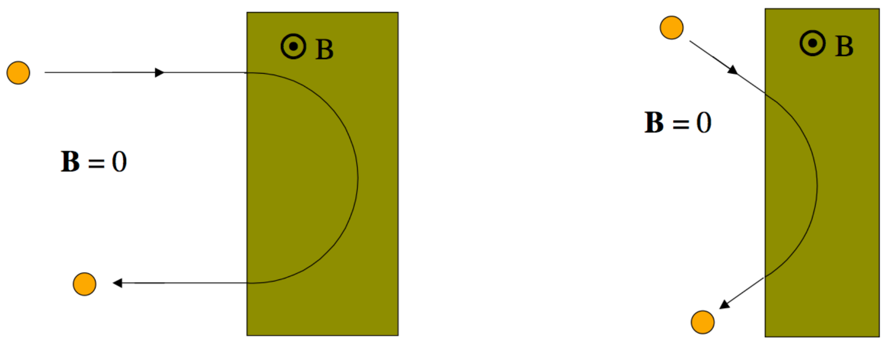

Fig. 3.1: Idealized reflexion on a magnetic cloud. The trajectory of the particle is rectilinear outside of the cloud (where the magnetic field is assumed to be 0) and circular inside the cloud (where the magnetic field is non 0 and perpendicular to the cloud plane). In this idealized version the reflexion angle is equal to the incidence angle.

3.2 The original Fermi mechanism (1949)

The original idea proposed by Fermi in 1949 for cosmic-ray acceleration was based on the fact that the interstellar medium is filled with "clouds" of ionized gas in movement with respect to the "Galactic frame". These clouds are carrying a magnetic field[1] and can in principle reflect the incoming charged particles (see Fig. 3.1). The acceleration mechanism based on moving "magnetic clouds" can be understood with a trivial but meaningful analogy with an idealized tennis game.

Let us assume a tennis ball is thrown, with a velocity v, on a steady racket. Ignoring the possible heat dissipation during the shock, the ball is simply reflected with the same velocity and no net energy gain, as it would be on a wall (see Fig. 3.2). Let us now assume that the racket is moving with a velocity V toward the ball (which still has a velocity v with respect to the tennis court). **In the racket frame**, the ball has a velocity $v + V$ and, assuming a perfectly elastic shock, the ball is reflected with the same (but opposite) velocity. **Back to the court frame** (adding V to the ball velocity in the racket frame), after the shock the ball has been accelerated to a velocity $v + 2V$ due to the head-on collision between the ball and the racket (see Fig. 3.3a). This result has been obtained with a double change of reference frame : court frame → racket frame → court frame.
Let us now consider a *dropshot*, meaning that the racket is now going away from the ball with a velocity V with respect to the court frame (see Fig. 3.3b). With the exact same calculation we conclude that the ball has been decelerated to a velocity $v - 2V$, in the court frame after the shock.

These simplistic analogies are enough to catch the essential origin idea behind Fermi's acceleration mechanism : one simply has to replace the tennis ball by a charged particle (a cosmic-ray) and the racket by a "magnetic cloud". Particles will be accelerated by each encounter with a magnetic cloud coming toward them and decelerated by the encounters with magnetic clouds going away from them. The energy gain (or loss) for each encounter can be calculated by a double change of reference frame, Galactic frame → cloud frame → Galactic frame.

[1] The magnetic field is turbulent due to the turbulent motion of the ionized gas but we will ignore this complication at the beginning and consider a simple configuration.

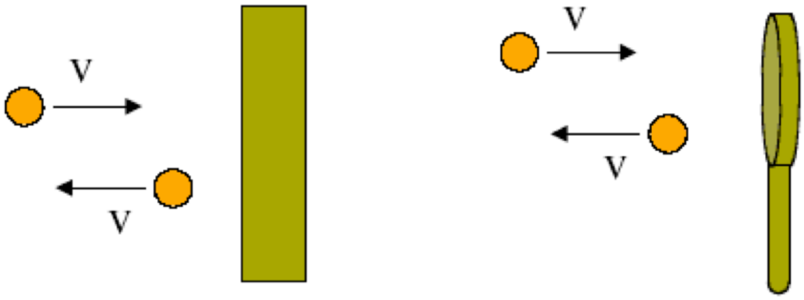

Fig. 3.2: Elastic reflexion of a ball by a wall or a tennis racket. The velocity is conserved during the shock, there is no net change in energy.

Fig. 3.3: The elastic reflexion of a tennis ball on a moving racket conserves the velocity (and then the energy) of the ball in the racket frame, it implies an energy gain in the court frame is the racket is moving toward the ball (left panel) or an energy loss if the racket is moving away from the ball (right panel).

Note that in this scheme the magnetic field is only the agent of the charged particles reflexion by the moving cloud. In the absence of magnetic field in the cloud the particles would just go through the moving cloud (as if the ball was going through the tennis racket) without any energy change (ignoring interactions with the cloud particles). We then expect the energy gain we will calculate in the following for some idealized cases to be independent of the magnetic field.

As mentioned above, another possible way of working out the acceleration the to calculate the electric field seen in the Galactic frame by Lorentz transformation of the pure B field seen in the cloud frame. Since the two approaches must be equivalent the acceleration and the energy gain of the particle must also be independent of the cloud magnetic field in this case. This result is however far less intuitive with this approach.

3.2.1 Calculation of the energy gain for an idealized single particle/cloud encounter

Let us consider the case of an idealized reflexion (for which the reflexion angle is equal to the incidence angle, see Figs. 3.1 and 3.4) of a particle (with a velocity $\vec{v}$) by a cloud coming toward it with a velocity $\vec{V}$, the incidence angle θ is then given by $\vec{V}.\vec{v} = -\cos\theta$. Moreover for typical

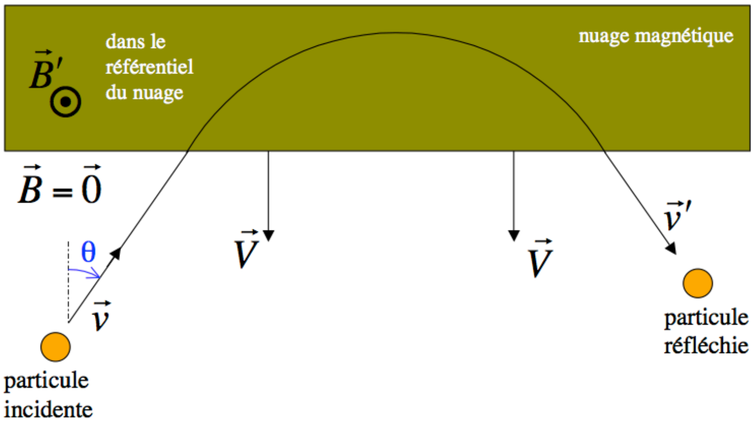

FIG. 3.4: Reflexion of a (positively) charged particle on a moving magnetic cloud.

Galactic magnetic clouds we have $V \ll v$ and $V \ll c$.

Calculation with a double change of reference frame

We use as a convention, primed quantities for the cloud frame and unprimed for the Galactic frame. Passing from the Galactic frame to the cloud frame we have :

$$\left\{ \begin{array}{l} E'_{in} = \gamma_{cloud}(E_{in} - \vec{P_{in}}.\vec{V}) = \gamma_{cloud}(E_{in} - P^{\|}_{in}.V) \\ P'^{\|}_{in} = \gamma_{cloud}(P^{\|}_{in} - \frac{V}{c^2} E_{in}) \end{array} \right. \tag{3.1}$$

where γ_{cloud} is the Lorentz factor of the cloud in the Galactic frame, the subscript *in* refers to the properties of the incoming particle and the parallel symbol "$\|$" refers to the projection along the cloud velocity vector.

Inside the cloud, we assume (and it is in principle a good approximation) the particle is just reflected and does not loose or gain energy so we have $E'_{out} = E'_{in}$, where the subscript *out* refers to the properties of the outgoing particle. Moreover, we assumed the encounter with the cloud leads to a perfect reflexion so we have $P'^{\|}_{out} = -P'^{\|}_{in}$.

We now come back to the Galactic frame with the inverse Lorentz transformation :

$$\left\{ \begin{array}{l} E_{out} = \gamma_{cloud}(E'_{out} + P'^{\|}_{out}.V) \\ P^{\|}_{out} = \gamma_{cloud}(P'^{\|}_{out} + \frac{V}{c^2} E'_{out}) \end{array} \right. \tag{3.2}$$

by substitution we get :

$$E_{out} = \gamma^2_{cloud} \left[E_{in}(1 + \frac{V^2}{c^2}) - 2P^{\|}_{in}.V \right] \tag{3.3}$$

and since $P^{\|}_{in} = -\frac{E_{in}.v \cos\theta}{c^2}$ (the minus sign comes from the above definition of θ) to first order in V/c we get :

$$E_{out} = E_{in}\left(1 + \frac{2v.V\cos\theta}{c^2}\right) \Leftrightarrow \frac{\Delta E}{E} = \frac{E_{out} - E_{in}}{E_{in}} = -2\frac{\vec{v}.\vec{V}}{c^2} \tag{3.4}$$

This final result leads to the following conclusions :

- The energy gain is proportional to the initial energy ($\Delta E/E$ is independent of E).
- The energy gain is positive for a head-on collision ($\vec{v}.\vec{V} < 0$).
- The energy gain is independent of B', as anticipated, the magnetic field mediates the reflection but it does not appear in the Lorentz transformations.

Alternative approach with the induced electric field

We place ourselves in the same configuration as before as schematized in Fig. 3.5. The electromagnetic tensor in the cloud frame can be transformed to the Galactic frame to obtain the electric field $\vec{E}$:

$$\vec{E_\perp} = \gamma_{cloud}(\vec{E'_\perp} - \vec{V} \times \vec{B'_\perp}) \tag{3.5}$$

since $E' = 0$ by hypothesis and to first order in V/c we get :

$$\vec{E_\perp} = -\vec{V} \times \vec{B'_\perp} \tag{3.6}$$

with this calculation of the electric field in the Galactic frame we can calculate the energy gain during the encounter :

$$\Delta E = \int_{\text{inside the cloud}} \vec{F}.\vec{dl} = q\vec{E_\perp}.\vec{dl} = qE_\perp l_{IJ} \tag{3.7}$$

the segment IJ being the projection of the circular trajectory along the electric field vector (see Fig. 3.5) and $q = Ze$ the charge of the particle. We have :

$$l_{IJ} = 2r_L \sin(\frac{\pi}{2} - \theta) = 2r_L\cos\theta \tag{3.8}$$

where $r_L = \frac{P}{ZeB} = \frac{E.v}{ZeBc^2}$ is the Larmor radius or the particle with energie E, momentum P, velocity v and charge Ze in the magnetic field B.
We finally get :

$$\Delta E = ZeVB \times 2\frac{Ev}{ZeBc^2}\cos\theta = \frac{2EVv\cos\theta}{c^2} = -2E\frac{\vec{v}.\vec{V}}{c^2} \Leftrightarrow \frac{\Delta E}{E} = -2\frac{\vec{v}.\vec{V}}{c^2} \tag{3.9}$$

This results is equivalent to that obtained in Eq. 3.4, with the method of the double change of reference frame. We also find that the magnetic field B is eliminated from the equation which is a lot less obvious in principle than when the calculation is performed by double change of reference frame. In the following will only consider the latter method which leads to much simpler calculations although it takes no account of the induced electric field which is physically the true responsible for the particles energy gains.

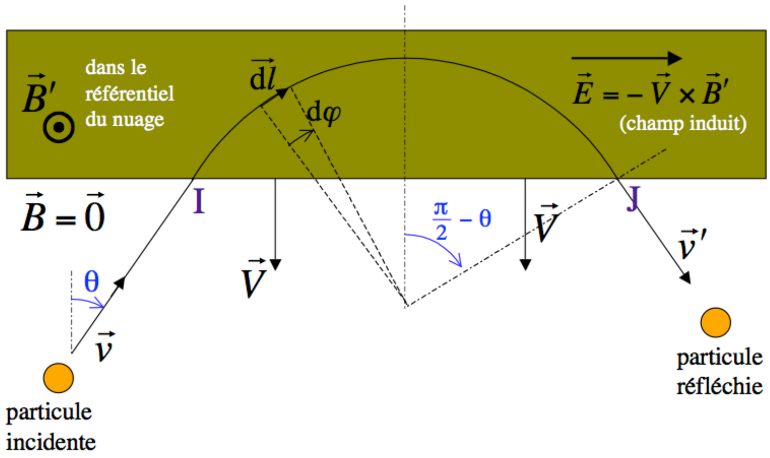

Fig. 3.5: Action of the induced electric field during the reflexion of a charged particle by a moving magnetic cloud.

3.2.2 Fuller calculation

So far we have studied single encounters with a cloud in the case of an idealized perfect reflexion. We saw that, indeed, a head-on collision was leading to an energy gain but also that energy can be lost if the cloud is moving away from the particle. The essence of Fermi's acceleration mechanism by interactions with moving Galactic magnetic clouds is that charged particles will suffer a series of encounter while propagating in the interstellar medium. Head-on collisions will lead to energy gains while energy will be lost when the cloud is moving away. All the point (which might not seem so intuitive at first sight) is that head-on collisions are **in average** more frequent so that in average particles do gain energy with this mechanism.

Why are head-on collisions more frequent? To understand that we can make a simple and familiar analogy with a car cruising on the highway. The car obviously crosses more cars coming toward than it overtakes cars going in the same direction. This is all a question of relative velocity and the same will be true with magnetic clouds. Moreover, on the highway the lower the velocity of the other cars the lower will be the difference between the number of car crossed and the number of cars overtaken. The same will be true for the magnetic clouds.

Calculation

We now relax our simplifying hypothesis of a perfect reflexion and place ourselves in the case of Fig. 3.6a. We now assume that the magnetic field is turbulent so that charged particles are isotropized inside the cloud and that the angle of the particle escaping from the cloud θ'_{out} is random (the subscript "1" refers to our subscript "in" in Fig. 3.6a). To simplify our derivations we now assume that the particles are already ultra-relativistic, *i.e* $E \simeq Pc$. The double change of reference frame gives :

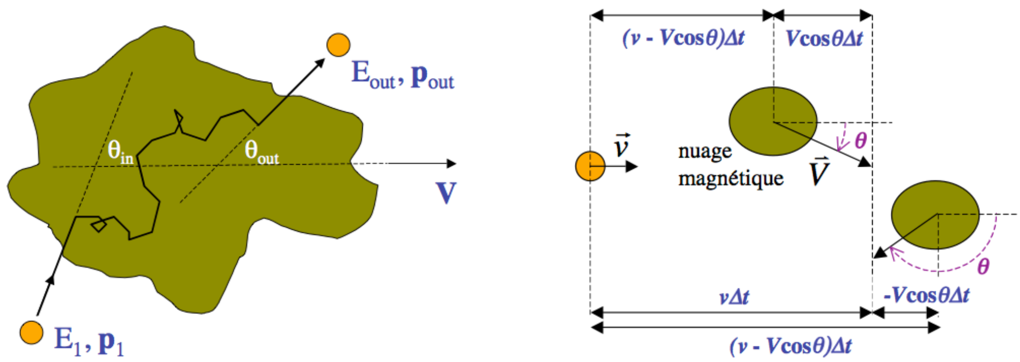

FIG. 3.6: Left : Schematic view of the "interaction" of a charged particle with a magnetic cloud. The particle enters the cloud and is isotropized by the magnetic turbulence. Right : Schematic view explaining the distribution of the incidence angle of particles with magnetic clouds : if the clouds velocity distribution is isotropic in the ISM and the density of cloud making an angle θ with the particle velocity vector is uniform the average number of clouds encountered during a time interval Δt is proportional to $(v - V\cos\theta)$.

$$\begin{cases} E'_{in} = \gamma_{cloud} E_{in}(1 - \beta_{cloud}\cos\theta_{in}) \\ E_{out} = \gamma_{cloud} E'_{out}(1 + \beta_{cloud}\cos\theta'_{out}) \end{cases} \tag{3.10}$$

using $E'_{in} = E'_{out}$ and dropping the subscript "cloud" we get :

$$E_{out} = \gamma^2 E_{in}(1 - \beta\cos\theta_{in})(1 + \beta\cos\theta'_{out}) \tag{3.11}$$

$$\Leftrightarrow \frac{\Delta E}{E} = \frac{E_{out} - E_{in}}{E_{in}} = \frac{\beta^2 - \beta\cos\theta_{in} + \beta\cos\theta'_{out} - \beta^2\cos\theta_{in}\cos\theta'_{out}}{1 - \beta^2} \tag{3.12}$$

to get the average energy gain, we need to average the above expression. By hypothesis, the particles are isotropized in the cloud, hence $\langle\cos\theta'_{out}\rangle = 0$. We now need to calculate $\langle\cos\theta_{in}\rangle$, the probability to have an encounter with an incidence angle θ_{in} should be proportional to the relative velocity between the particles and the cloud (think of the car on the highway) in the case of uniformly distributed clouds. It gives (see Fig. 3.6b) : $P(\theta_{in}) \propto v - V\cos\theta_{in}$ ($v \simeq c$ and V still being respectively the velocity of the particle and the cloud). We then have :

$$\langle\cos\theta_{in}\rangle = \frac{\int_{-1}^{1}\cos\theta_{in}(v - V\cos\theta_{in})d\cos\theta_{in}}{\int_{-1}^{1}(v - V\cos\theta_{in})d\cos\theta_{in}} = \frac{-2V/3}{2v} \simeq \frac{-2V/3}{2c} \simeq -\frac{\beta}{3} \tag{3.13}$$

substituting in Eq. 3.12 we finally get :

$$\langle\frac{\Delta E}{E}\rangle = \frac{\beta^2 + \beta^2/3}{1 - \beta^2} \simeq \frac{4\beta^2}{3} \tag{3.14}$$

We get an average energy gain which is indeed positive hence Fermi's mechanism is truly an acceleration mechanism for charged particles. The average fractional energy gain is proportional to

β^2 so this mechanism is often called *second order Fermi mechanism.* This mechanism is stochastic by nature, the average energy gain is positive but for a given particle, the energy gain can vary depending on the configuration of the series of encounters.

Acceleration time

The fact that the mechanism proposed by Fermi leads to a net energy gain in average does not mean that this mechanism is responsible for the production of the Galactic cosmic-rays we observe. In particular, to be a good candidate this mechanism must be sufficiently fast at accelerating particle to the very high energies measured by cosmic-ray observatories. To get a quantitative answer to this question, we need an approximate estimate of the particles acceleration times.

We start by giving the definition of the acceleration time t_{acc} :

$$t_{acc}(E) = \left(\frac{1}{E}\frac{dE}{dt}\right)^{-1} \tag{3.15}$$

Let L be the typical distance between two clouds and let us assume there is no magnetic field between two cloud (our estimate then becomes a lower limit), then the average time between two encounters will be $\langle t_{coll} \rangle = \frac{L}{c}$, neglecting the time spent by the particles inside the cloud we then get :

$$\frac{dE}{dt} \simeq \frac{\Delta E}{\langle t_{coll} \rangle} = \frac{4}{3}\frac{\beta^2 cE}{L} = \frac{E}{t_{acc}} \Leftrightarrow t_{acc} = \frac{3}{4}\frac{L}{c}\beta^{-2} \tag{3.16}$$

Power law spectrum

Let us consider the acceleration time t_{acc}, defined in Eq. 3.15, and the escape time t_{esc}. The latter would be the average time for the particles to leave the region of the Galaxy where the magnetic clouds are present. Let us assume, to simplify the calculation that t_{acc} and t_{esc} are both independent of the energy. It would make sense for our simple model since we have assumed that the medium was not magnetized and neglected the duration of the interaction between particles and clouds. Assuming particles are injected at $t = 0$ at an energy E_0 then :

$$E(t) = E_0 \exp\left(\frac{t}{t_{acc}}\right) \tag{3.17}$$

it means that particles with an energy E have been in the acceleration site for a time $t(E) = t_{acc}\ln(E/E_0)$ and that particles with energy between E and $E + dE$ have been injected at a time between $t(E)$ and $t(E + dE) = t(E) + dt$ earlier, with $dt = \frac{dt}{dE}dE = \frac{t_{acc}}{E}dE$ (using the definition of t_{acc}).

Moreover, during a time dt, the escape probability is $P_{esc} = \frac{dt}{t_{esc}}$. Let us call $\dot{N}_0$, the injection rate, the amount of particles injected during the time interval dt is $\dot{N}_0 dt$. Among these particles, only a fraction $f = \exp\left(-\frac{t(E)}{t_{esc}}\right)$ are still in the system after a time $t(E)$ and are able to reach an energy E. So the number of particles between E and $E + dE$ is :

$$n(E)dE = \dot{N}_0 dt \exp\left(-\frac{t(E)}{t_{esc}}\right) dE = \dot{N}_0 \frac{t_{acc}}{E} \exp\left(-\frac{t_{acc}\ln(E/E_0)}{t_{esc}}\right) dE \tag{3.18}$$

and we then get

$$n(E) = \frac{\dot{N}_0}{E_0} t_{acc} \left(\frac{E}{E_0}\right)^{-x} \tag{3.19}$$

with,

$$x = 1 + \frac{t_{acc}}{t_{esc}} \tag{3.20}$$

We see that we indeed obtained a power law shape for the spectrum of accelerated particles, which is in principle a good point in favor of this mechanism. However, the exact shape of the spectrum depends on the t_{acc}/t_{esc} which cannot be predicted in principle since it will depend on the exact configuration of the region where one can find a large concentration of magnetic clouds. It is moreover likely that different regions in the Galaxy would accelerate cosmic-ray with different power law shape and the sum of the contribution would not be likely to give a global power law as observed for Galactic cosmic-rays detected on Earth.

Order of magnitude calculation and conclusions

We can now apply our results to the case of the Galactic interstellar medium where Fermi's magnetic cloud are found. The typical velocity of a cloud is $\beta_{cloud} \simeq 10^{-4}$ which means $\beta^2_{cloud} \simeq 10^{-8}$ and the typical distance $L \simeq 1pc$. Using these number one rapidly understands that with this acceleration mechanism it would take almost a billion year for a particle to double its energy. This is way too long to reach the very high energies observed for Galactic cosmic-rays. Moreover, we have neglected energy losses which might take place in the interstellar medium (such as ionization losses or spallation) and it turns out that for GeV nuclei, for instance, the energy loss time in the interstellar medium would be shorter than the acceleration time we just calculated.

We must conclude that the original acceleration mechanism presented by Fermi is far too slow to account for cosmic-ray acceleration (at least as the dominant process). We must not however forget the vertues of this pioneering idea, this mechanism indeed leads, in principle, in average to a net energy gain for the accelerated particle. Moreover although this scenario predicts a power law shape for the spectrum of accelerated particles, a robust prediction of the shape of the spectrum is very difficult to obtain. Despite its failure, Fermi mechanism is in fact the seed of most of the modern acceleration mechanisms which have been proposed since Fermi's pioneering work. Diffusive shock we will study in the next section, directly inherit of the basic ideas we discussed in this section, we will see that this mechanism manages to overcome quite elegantly most of the problems of the original Fermi mechanism.

3.3 Diffusive shock acceleration (DSA)

In the previous section, we saw that Fermi proposed in 1949 an acceleration mechanism which works in principle but turns out to be inefficient when considering realistic cases. The failure of this mechanism mainly due to the fact that the fractional energy gain is proportional to β^2. Unless the velocity of the clouds was very much higher (in which case however some of the approximations we made in our calculations would not be valid anymore), the excess of the number of head-on collisions remains tiny. One could show that, if we had considered only head on collisions, the energy gain would have been proportional to β, rather than β^2. A mechanism which would provide only head-on collisions between particles and "clouds" would be much more likely to be a good candidate mechanism for cosmic-ray acceleration.

In this section, we will see that "astrophysical shocks" provide in principle such a mechanism, the so-called *"Diffusive Shock Acceleration" (DSA)*. Astrophysical shocks exist everywhere in the universe, from the solar system to more extreme objects such as supernova remnants, active galactic nuclei or gamma-ray bursts. We furthermore know particles are accelerated at these shocks as we detect radiation (from radio frequencies to γ-rays) from these object most often interpreted as the result of energy losses of accelerated electrons (see next chapter).

Astrophysical shock waves obey the same macroscopic rules as shock waves on Earth. They originate from outflows propagating with velocities larger than the local speed of sound. A shock front, through which physical quantities are discontinuous[2], forms. However, unlike "terrestrial shocks", the main difference is that astrophysical shock waves are in most cases *collisionless*, *i.e* the shock and the energy dissipation processes do not take place through particles collisions or coulombian interactions. The shock formation is due to the "interaction" of particles with the ambient magnetic field. Without the fields, supersonic outflows would just pass "unnoticed" through the ambient medium without the formation of any shock. Beyond these very simple considerations, the microphysics of astrophysical shocks is extremely complicated and far beyond the scope of this course.

3.3.1 Jump conditions

When a shock wave propagates through a medium, for instance the interstellar medium, one must distinguish between the *downstream* ("aval" in French, which is the region which has already been shocked) and the *upstream* ("amont" in French, which is the region of space ahead of the shock that has not been shocked yet) regions. Jump condition which relate physical quantities in the upstream and downstream can be obtained by writing down different conservation laws (matter, momentum and energy conservation) when passing through the shock (see Fig. 3.7).

Let us place ourselves in the shock front frame, we use the index "1" for physical quantities upstream and the index "2" for the physical quantities downstream. The density the pressure, the temperature and the flow velocity in the two media (ρ_i, P_i, T_i, v_i) are determined by the conservation relation at the shock. The three thermodynamical conservation equations which determine the relation between downstream and upstream quantities are :

- Mass conservation

$$\rho_2 v_2 = \rho_1 v_1 \tag{3.21}$$

- Momentum flux conservation

$$P_2 + \rho_2 v_2 = P1 + \rho_1 v_1 \tag{3.22}$$

[2]More precisely, they vary abruptly on a very small scale

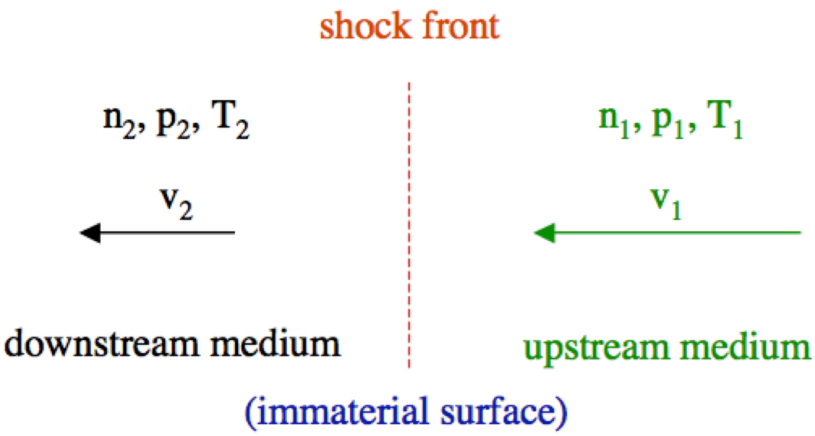

Fig. 3.7: Schematic view of a shock wave propagating in a medium, as seen in the shock rest frame. The shock is at rest and the upstream medium is coming toward it with a velocity v_1 while the downstream medium is going away with a velocity v_2. Physical quantities are discontinuous through the (immaterial) surface of the shock.

- Energy conservation

$$\rho_2 v_2 \left(v_2^2 + \frac{p_2}{\rho_2} + e_2 \right) = \rho_1 v_1 \left(v_1^2 + \frac{p_1}{\rho_1} + e_1 \right) \tag{3.23}$$

where e_i is the energy density given by $e_i = \frac{1}{\gamma_a - 1}\frac{P_i}{\rho_i}$ for a perfect gas and γ_a is the adiabatic index (warning : do not confuse with the Lorentz factor) of the gas ($\gamma_a = 5/3$ for a monoatomic gas).

The solutions of these conservation equations are for the flow velocities :

$$\frac{v_2}{v_1} = \frac{\gamma_a + M_1^{-2} \pm (1 - M_1^{-2})}{\gamma_a + 1} \tag{3.24}$$

where we have introduced the Mach number $M_1 = \frac{v_1}{v_{1,\,sound}}$ which is the ratio of the shock velocity in the upstream medium to the local sound velocity.

As can be seen, there are two possible solution. The "+" yields $v_2 = v_1$ which correspond to a uniform flow without discontinuity. The "-" solution correspond to a shock with a discontinuity of the velocity at the shock :

$$\frac{v_2}{v_1} = \frac{\gamma_a - 1 + 2M_1^{-2}}{\gamma_a + 1} \tag{3.25}$$

The other quantities can be derived as well (see the chapter 11 of Longair's "High energy astrophysics") :

$$\frac{\rho_2}{\rho_1} = \frac{\gamma_a + 1}{\gamma_a - 1 + 2M_1^{-2}} \tag{3.26}$$

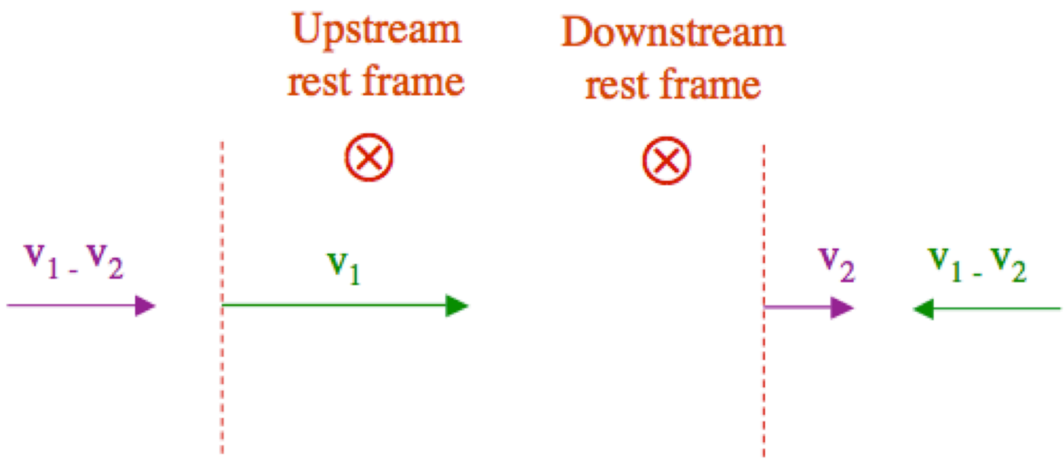

Fig. 3.8: Schematic view of the shock wave as represented in Fig. 3.7, this time seen in the upstream (left) and downstream (right) media rest frames.

$$\frac{P_2}{P_1} = \frac{2\gamma_a M_1^2 - (\gamma_a - 1)}{\gamma_a + 1} \tag{3.27}$$

$$\frac{T_2}{T_1} = \frac{\left[2\gamma_a M_1^2 - (\gamma_a - 1)\right]\left[\gamma_a - 1 + 2M_1^{-2}\right]}{(\gamma_a + 1)^2} \tag{3.28}$$

When passing through the shock, the fluid is compressed, heated and slows down ($\rho_2 > \rho_1$, $T_2 > T_1$, $P_2 > P_1$, $v_2 < v_1$). We can define the compression ratio r such that $\rho_2 = r\rho_1$, we then also have $v_2 = \frac{v_1}{r}$. In the limit of strong shocks, we have $M_1 \gg 1$ and $r \simeq \frac{\gamma_a+1}{\gamma_a-1}$, for a monoatomic gas, $\gamma_a = 5/3$ and as a consequence, $r = 4$.

3.3.2 Principle of DSA

If we concentrate only on the velocity discontinuity, we can easily understand the interest of shock waves for particles acceleration. In the shock frame, the upstream medium is coming toward the shock with a velocity v_1 (note of course that $v_1 = v_{sh}$ where v_{sh} is the shock velocity). Passing through the shock the gas slows down and the downstream medium is moving away from the shock with a velocity $v_2 = v_1/r = v_{sh}/r$ (see Fig. 3.7).

Let us now consider an observer at rest in the upstream frame. He sees the shock approaching with a velocity $v_1 = v_{sh}$, he also sees the downstream medium approaching with a velocity[3] $\Delta v = v_1 - v_2 = \left(\frac{r-1}{r}\right) v_{sh}$ (see Fig. 3.8a).

Now for an observer at rest with respect to the downstream fluid, the shock is going away with the velocity v_2 we obtained before, but the upstream medium is coming toward the observer, again with a velocity $\Delta v = v_1 - v_2 = \left(\frac{r-1}{r}\right) v_{sh}$ (see Fig. 3.8b).

[3]Note that we use the non-relativistic velocity composition which mean we assume that the shock velocity $v_{sh} \ll c$.

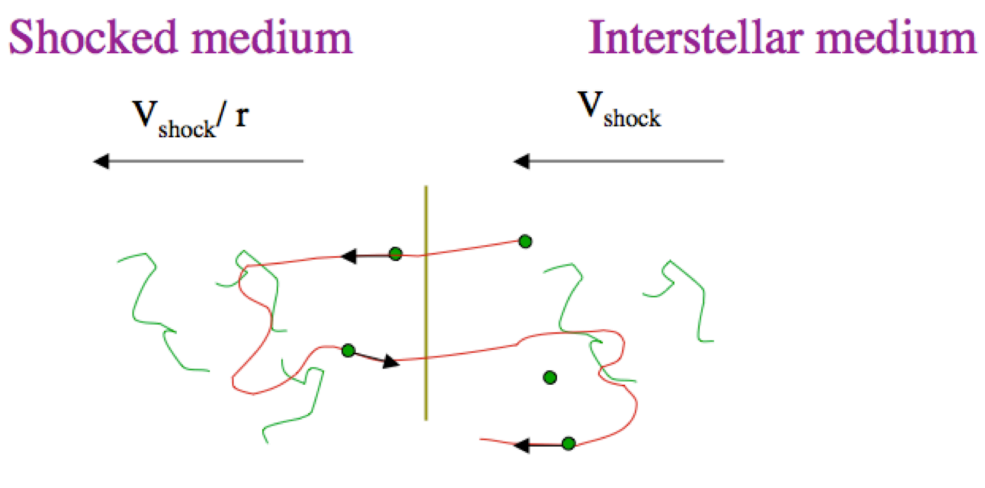

FIG. 3.9: Schematic view of a cycle as seen in the shock rest frame : the particle initially in the ISM (upstream medium) enters the shocked medium (downstream medium), it is then isotropized by the magnetic turbulence and reflected back to the upstream medium where the particle is isotropized again and eventually crossed the shock to start a new cycle.

Let us assume the both the upstream and downstream media are magnetized [4]. We are then in a situation where a particle coming from the upstream medium and passing through the shock would see the downstream medium as a "magnetic cloud" coming toward it (a cloud with a velocity Δv with respect to the upstream fluid rest frame). Likewise a particle coming from the downstream medium and passing through the shock would see the upstream medium as a "magnetic cloud" coming toward it (with a velocity Δv with respect to the downstream fluid rest frame). We can then understand that a particle which would cross several times the shock for instance from upstream to downstream then back upstream could gain energy by interacting with moving "magnetic clouds". The critical difference with the original mechanism proposed by Fermi is that with this configuration, all the collisions would now be head-on. It is then likely that this mechanism involving charged particles cycling across a shock front will turn out to be much more efficient than the original mechanism proposed by Fermi. This prove this statement we need to perform the calculation of the energy variation experienced by a charged particle during a cycle *upstream* $\rightarrow$ *downstream* $\rightarrow$ *upstream*.

3.3.3 Energy gain after a cycle (upstream $\rightarrow$ downstream $\rightarrow$ upstream)

Before calculating the mean energy gain experienced by a charged particles during a cycle, we need to set a series of physical hypotheses to define the framework of our calculation. The relevance of these hypothesis will be discussed later on, throughout this section :

- The shock is an infinite plane
- The media upstream and downstream of the shock have an infinite spatial extension

[4] and we have good reasons to make this assumption since we know that the undisturbed interstellar medium is magnetized.

- There is no limitation in time, the relevant physical quantities are in a steady state
- There are magnetic field inhomogeneities, both upstream and downstream of the shock which "isotropize" energetic charged particles, the propagation of particles is diffusive in both media.
- The shock is non-relativistic $v_{sh} \ll c$, the charged particles are relativistic $v_{part} \simeq c \gg v_{sh}$.

For this calculation we use unprimed quantities for the upstream frame and primed quantities for the downstream frame. Let θ_{in} be the the angle between the particle velocity and the shock normal at the initial shock crossing in the upstream frame and θ'_{out} the angle of the particle with the shock normal in the downstream frame, when crossing the shock back to the upstream medium. On a cycle upstream → downstream → upstream, we have :

$$\begin{cases} E'_{in} = \gamma E_{in}(1 - \beta \cos\theta_{in}) \\ E_{out} = \gamma E'_{out}(1 + \beta \cos\theta'_{out}) \end{cases} \tag{3.29}$$

γ and β would correspond to γ_{cloud} and β_{cloud} in the original Fermi mechanism. In the case of a shock wave, they correspond to the velocity of the downstream medium in the upstream fluid frame (see above) and then we have $\beta = \Delta v/c$ and $\gamma = \frac{1}{\sqrt{1-\frac{\Delta v^2}{c^2}}}$. Using $E'_{in} = E'_{out}$, we get the already familiar expression :

$$\frac{\Delta E}{E} = \frac{\beta^2 - \beta\cos\theta_{in} + \beta\cos\theta'_{out} - \beta^2\cos\theta_{in}\cos\theta'_{out}}{1-\beta^2} \tag{3.30}$$

again, the mean fractional energy gain is obtained by averaging $\langle\cos\theta_{in}\rangle$ and $\langle\cos\theta'_{out}\rangle$, the result will however differ from the original case with magnetic clouds.

To obtain $\langle\cos\theta_{in}\rangle$ and $\langle\cos\theta'_{out}\rangle$, we need to know the probability of crossing the shock with an angle between θ and $\theta + d\theta$. Let us neglect the shock velocity since $v_{sh} \ll v_{part}$. The particle crosses the shock with a velocity $v_{part}\cos\theta$ and then asuming a particle density n_0, the number of particles passing the shock with an angle between θ and $\theta + d\theta$, through a surface dS during a time dt is :

$$dN^4 = \frac{n_0}{4\pi} v_{part}\cos\theta d\Omega dS dt = \frac{n_0}{2} v_{part}\cos\theta\sin\theta d\theta dS dt \tag{3.31}$$

the probability of crossing the shock with an angle between θ and $\theta + d\theta$ is then proportional to $\cos\theta\sin\theta d\theta$. We then have :

$$\langle\cos\theta\rangle = \frac{\int_{\theta_{min}}^{\theta_{max}}\cos^2\theta\sin\theta d\theta}{\int_{\theta_{min}}^{\theta_{max}}\cos\theta\sin\theta d\theta} = \frac{\left[\frac{1}{3}\cos^3\theta\right]_{\theta_{min}}^{\theta_{max}}}{\left[\frac{1}{2}\cos^2\theta\right]_{\theta_{min}}^{\theta_{max}}} \tag{3.32}$$

for the crossing from upstream to downstream we have $\theta_{min} = \frac{\pi}{2}$ and $\theta_{max} = \pi$ (particle and shock velocities are antiparallel) which give :

$$\langle\cos\theta_{in}\rangle = -\frac{2}{3} \tag{3.33}$$

for the crossing back from downstream to upstream we have $\theta_{min} = 0$ and $\theta_{max} = \frac{\pi}{2}$ which give :

$$\langle \cos \theta'_{out} \rangle = \frac{2}{3} \tag{3.34}$$

Note that we implicitly got rid of some complications and of second order corrections by neglecting the shock velocity in this calculation. Since $v_{sh} \ll c$ we also have $\Delta v \ll c$ we can then neglect terms in β^2 in Eq. 3.30 calculating $\langle \Delta E/E \rangle$. We finally get :

$$\langle \frac{\Delta E}{E} \rangle = \frac{4}{3}\beta = \frac{4}{3}\beta_{sh}\left(\frac{r-1}{r}\right) \tag{3.35}$$

where $\beta_{sh} = v_{sh}/c$ and r is the above-mentioned compression ratio.

As anticipated, DSA is indeed an acceleration mechanism (since $\langle \frac{\Delta E}{E} \rangle$ is positive) with an energy gain proportional to β which is a very important step forward when comparing with the original Fermi mechanism. DSA is also called **first order Fermi mechanism** for obvious reasons.

3.3.4 Spectrum of accelerated particles

After calculating the mean energy gain obtained after a single cycle upstream $\rightarrow$ downstream $\rightarrow$ upstream, one can try to estimate the number of cycles this particles are going to achieve before "leaving the system". At first place, it is important to understand what mechanism is going to limit the number of cycles a particle can perform. The critical point is that during each cycle a given has a probability not to come back to the shock, in other word a probability to escape from "the acceleration region".

Within our hypothesis (infinite media upstream and downstream and steady state), the particles have no way of escaping upstream of the shock. Indeed since by hypothesis the accelerated particles are isotropized by the ambient magnetic fields the accelerated particles fluid has no net velocity with respect to the medium (either downstream or upstream) rest frame. As in the upstream medium frame the shock is "going after the particles", the return probability at the shock is 1, meaning that the escape probability is 0. In the downstream medium on the other hand, the cosmic-ray fluid has a zero velocity with respect to the ambient medium and the shock is going away with a velocity $v_2 = v_{sh}/r$. This means that the accelerated particles fluid is in average slowly advected away from the shock. The escape probability can be estimated by comparing the flux of particles being advected far away from the shock with the flux of particle entering the downstream medium by crossing the shock from the upstream medium. Before performing this trivial calculation, we can give an alternative reasoning to understand the impossibility of leaving the system in the upstream frame.

Let us now consider individual particles rather than the "accelerated particles fluid" and let's think in term of diffusion as we did in the chapter dedicated to Galactic cosmic-rays. Let us assume a particle enters the upstream medium (crossing the shock from upstream) at $t = 0$. Since we have an infinite amount of time available and since the upstream medium is infinite (at least in the framework of our hypotheses) diffusion theory tells us that the probability for the particle to come back to the plane where it crossed the shock within the infinite amount of time available is 1. Then the probability to cross the shock again would be 1 even if the shock was not moving. The fact that the shock is going after the particle in the upstream medium makes it even easier for the particle to cross the shock again.

Downstream, the probability for the particle to come back to the plane where it crossed the shock (from upstream) is still one, but the problem is that during the time it took for the particle to come back, the shock has moved away. If one calculated, using diffusion theory, the probability for a particle, not only to come back to where it last crossed the shock, but where the shock actually is at a given time t between 0 and ∞ (which is actually the condition for the particle to come back to the shock and be able to cross it again and which is, unlike in the upstream medium, a more stringent condition than just coming where it last crossed the shock) then one would find a probability smaller than 1, as a result of the shock going away with respect to the downstream fluid frame.

Let us now calculate the escape probability using our first reasoning in terms of accelerated particles fluid. Let n_0 be the accelerated particle density. Due to the global advection of the downstream fluid away from the shock front with a velocity v_2, the flux of accelerated particles passing through a unit surface very far away from the shock is $\phi_{esc} = n_0 v_2$. On the other hand the flux of particles crossing the shock from upstream to downstream is given by :

$$\phi_{ud} = \frac{n_0}{2} v_{part} \int_{\pi/2}^{\pi} \cos\theta \sin\theta d\theta = \frac{n_0}{4} v_{part} \simeq \frac{n_0}{4} c \tag{3.36}$$

The escape probability is then simply the ratio of the two fluxes :

$$P_{esc} = \frac{\phi_{esc}}{\phi_{ud}} \simeq \frac{4}{r} \beta_{sh} \tag{3.37}$$

We then have obtained so far $\langle \Delta E \rangle = \frac{4}{3} \left(\frac{r-1}{r} \right) \beta_{sh} E = kE$ and $P_{esc} = \frac{4}{r} \beta_{sh}$. We have everything we need to predict the slope of the accelerated particle spectrum :

After 1 cycle, the mean energy of particles injected with the energy E_0 is :

$$E_1 = (1+k)E_0 \tag{3.38}$$

thus after n cycles :

$$E_n = (1+k)^n E_0 \Leftrightarrow n = \frac{\ln(E/E_0)}{\ln(1+k)} \tag{3.39}$$

on the other hand, if we initially injected N_0 particles then after n cycles we have :

$$N_n = N_0(1+P_{esc})^n = N_0(1-P_{esc})^{\frac{\ln(E/E_0)}{\ln(1+k)}} = N(\geq E_n) \tag{3.40}$$

the right hand side just meaning that particles which have achieved n end up with energies greater or equal to E_n (since they will either escape at the cycle or continue for more cycles). Using $a^{\ln b} = e^{\ln a \ln b} = b^{\ln a}$ and dropping the subscript n we get :

$$N(\geq E) = N_0 \left(\frac{E}{E_0} \right)^{\frac{\ln(1-P_{esc})}{\ln(1+k)}} \tag{3.41}$$

and since the shock is non-relativistic, $\beta_{sh} \ll 1 \Leftrightarrow P_{esc} \ll 1$ and $k \ll 1$:

$$N(\geq E) = N_0 \left(\frac{E}{E_0}\right)^{\frac{-P_{esc}}{k}} \tag{3.42}$$

moreover

$$N(\geq E) = \int_E^{+\infty} n(E)dE \tag{3.43}$$

where $n(E)dE$ is the number of particles with energy between E and $E + dE$ and is the quantity we are looking for. Then,

$$n(E) = \left|\frac{dN(\geq E)}{dE}\right| = (x-1)\frac{N_0}{E_0}\left(\frac{E}{E_0}\right)^{-x} \tag{3.44}$$

with (after developing P_{esc}/k),

$$x = \frac{r+2}{r-1} \tag{3.45}$$

We obtain a power law spectrum with a *spectral index* x depending only on the shock compression ratio r! For a monoatomic gas and a strong shock ($\gamma_a = 5/3$, $M_1 \gg 1$), the slope of the power law is $x = 2$ and is "universal"[5].

Note that in the previous section we also derived with an alternative reasoning the prediction of a power law shape of the spectrum for the original Fermi mechanism (see Eq. 3.20) : $x = 1 + \frac{t_{acc}}{t_{esc}}$. This result is more general than the simple demonstration we derived suggest. A power law is obtained if t_{acc} and t_{esc} are independent of the energy. Which is the case for the model of the original Fermi mechanism we presented earlier. Another possibility is that the ratio $\frac{t_{acc}}{t_{esc}}$ is independent of the energy[6], or in other words that t_{acc} and t_{esc} evolve the same way with the energy. In the case of DSA, we can understand qualitatively that this second possibility will turn out to be true. Indeed, t_{acc} is related to the time needed to achieve a cycle. It is obviously not independent of the energy since for a high energy particle it will take longer to be isotropized a then to come back to the shock than for a lower energy particle.

It is more subtle for the escape time, but one can understand that t_{esc} also depends on the energy and for the same reason : a high energy particle takes longer to be isotropized and as a result can explore a larger region downstream before we can consider that the particle has no chance to come back to the shock anymore. In other words, the ratio $\frac{t_{acc}}{t_{esc}}$ is independent of the energy because both processes (acceleration and escape) are diffusive by nature and both depend the same way on the diffusion coefficient $D(E)$.

3.3.5 Acceleration time

It can be shown in diffusion theory that the time needed to complete a cycle is given by :

[5]Let us however keep in mind that this calculation relies on some simplifications and physical assumptions.

[6]The demonstration is of course in this case slightly different.

$$\langle \Delta t_{cycle}(E) \rangle = 4 \left(\frac{D_1(E)}{v_1 c} + \frac{D_2(E)}{v_2 c} \right) \tag{3.46}$$

where D_1 and D_2 are the diffusion coefficients for the upstream and downstream media respectively. Then for the acceleration time we have :

$$\langle t_{acc}(E) \rangle = \frac{\langle \Delta t_{cycle}(E) \rangle}{\langle \frac{\Delta E}{E} \rangle} \propto \frac{D(E)}{v_{sh}^2} \tag{3.47}$$

The energy dependence of the acceleration time is then given by the energy dependence of the diffusion coefficient which is related to the type of magnetic turbulence. As we saw in Chapt. 2 that for a Kolmogorov turbulence, n the limit $r_L(E) \ll \lambda_{max}$ we have $D(E) \propto E^{1/3}$. We will make a different assumption in the following mostly for the sake of simplicity.

Indeed, in most calculation, the *Bohm scaling* (or Bohm approximation) is usually assumed for the energy evolution of the diffusion coefficient. In the general case : $D(E) = \frac{1}{3} l_{scat} v$ where l_{scat} is the scattering length (or mean free path), *i.e* the length on which the particle get "randomized" and losses the memory of its initial direction and v is the velocity of the particle. The Bohm diffusion coefficient is obtained by assuming $l_{scat}(R) = r_L(E)$, where $r_L = \frac{P}{ZeB} = \frac{R}{Bc}$ is the Larmor radius of a particle of momentum P and rigidity R[7] (see Chapt. 2). We then have,

$$D_{Bohm}(R) = \frac{1}{3} r_L(R) c \tag{3.48}$$

With the Bohm hypothesis, we then get :

$$\langle t_{acc}(R) \rangle \propto \frac{R}{B} \times \frac{1}{v_{sh}^2} \tag{3.49}$$

(a dependence in $R^{1/3}$ would be expected for a Kolmogorov turbulence).

Before going further we can obtain some useful scaling laws (the reader is invited to recalculate them) :

$$r_L(R) \simeq 1.1 \times \left(\frac{R}{10^{15}\mathrm{V}} \right) \times \left(\frac{B}{\mu\mathrm{G}} \right)^{-1} \mathrm{pc} \tag{3.50}$$

$$D_{Bohm}(R) \simeq 3.4 \times 10^{28} \times \left(\frac{R}{10^{15}\mathrm{V}} \right) \times \left(\frac{B}{\mu\mathrm{G}} \right)^{-1} \mathrm{cm^2\,s^{-1}} \tag{3.51}$$

Let us now quantify a little our finding using typical parameters probably at play in a supernova remnant. We remind that *supernova remnants* are what remains of a supernova hundreds to tens of thousands of years after a supernova explosion. The phenomenon is explained by the propagation of a supersonic plasma, emitted at the time of the supernova event, through the interstellar medium. The shock wave is known to heat up the shocked interstellar medium and to accelerate electrons and nuclei which emit non thermal radiation up to very high energies. Supernova remnants are considered as one of the best source candidate for the acceleration of Galactic cosmic-rays[8]. The physical parameters which are important in order to discuss cosmic-ray acceleration by supernova

[7] We remind that $R \simeq E/Z$ for ultra-relativistic particles

[8] There is however no direct proof of this connection.

remnant evolve with time[9], we will limit ourselves for the type being to typical values thought to be at play in the vicinity of the shock wave a few hundred years after the supernova explosion.

The typical value of the magnetic field and the shock velocity is thought to vary from remnants to remnants $B \sim [100, 500]\,\mu\mathrm{G}$, $v_{sh} \sim [1000, 10000]\,\mathrm{km\,s^{-1}}$. Let us use $B = 100\ \mu\mathrm{G}$ and $v_{sh} = 3000$ $\mathrm{km\,s^{-1}}$ as typical values. For a proton at 100 GeV we have $D_{Bohm}(100GeV) \simeq 3.4 \times 10^{22}\,\mathrm{cm^2\,s^{-1}}$. Thus,

$$\langle \Delta t_{cycle} \rangle_{100\,\mathrm{GeV}} \simeq \frac{4D(E)}{cv_{sh}} \simeq 1.5\,10^4\,\mathrm{s} \tag{3.52}$$

$$\langle t_{acc} \rangle_{100\,\mathrm{GeV}} = \frac{\langle \Delta t_{cycle} \rangle_{100\mathrm{GeV}}}{\langle \frac{\Delta E}{E} \rangle} \simeq \frac{4D(E)}{v_{sh}^2} \simeq 1.5\,10^6\,\mathrm{s} \simeq 17.5\,\mathrm{days} \tag{3.53}$$

Using $t_{acc}(E) \propto E$ (assuming Bohm scaling) we get :

$$\langle t_{acc} \rangle_{10^{15}\,\mathrm{eV}} \simeq 480\,\mathrm{years} \tag{3.54}$$

Although the discussion of cosmic-ray acceleration in supernova remnants goes of course far beyond the order of magnitude calculation we just made, we can use our results to compare with what we obtained for the second order Fermi mechanism we discussed in the previous section. The comparison speaks for itself. With DSA we now get much more reasonable acceleration times which can be achieve within the source lifetime (see next chapter for a more complete (although oversimplified) discussion) and which can also compete with energy loss mechanisms.

3.3.6 Particle confinement in the source and maximum achievable energy

At this stage, it is useful to discuss the some of the assumptions we made to calculate the accelerated particle spectrum, *i.e* infinite media (upstream and downstream) and steady conditions (shock lasting forever).

In any astrophysical source, the acceleration region has in principle a finite spatial extension (and of course a finite age and activity duration) and as a consequence, the Larmor radius of the accelerated particles cannot exceed the size of the acceleration site.

Taking for instance $B = 100\ \mu G$ and $L_{source} = 1$ pc, one can estimate a maximum energy E_{max}^{size} such that $r_L(E_{max}^{size}) = L_{source}$. With these parameter, one gets $E_{max}^{size} \simeq Z \times 10^{17}$ eV or in terms of the maximum rigidity $R_{max}^{size} \simeq 10^{17}$ V.

An argument very close to that one was used by M. Hillas (in 1984) to build the famous *Hillas diagram* estimating the maximum energy achievable for cosmic-rays in different types of sources as a function of their size and their ambient magnetic field. Hillas constructed his famous diagram in order to select viable sources for cosmic-ray acceleration above 10^{20} eV, a version of it is visible in Fig. 3.10. We must note that the "Hillas condition" is a necessary but not sufficient condition to estimate whether or not a given energy can be achieve in a given type of source. In most case as we will discuss later the maximum energy estimated with this criterion is overoptimistic. Moreover

[9] The way they evolve with time is matter of intense debate, for the magnetic fields for instance

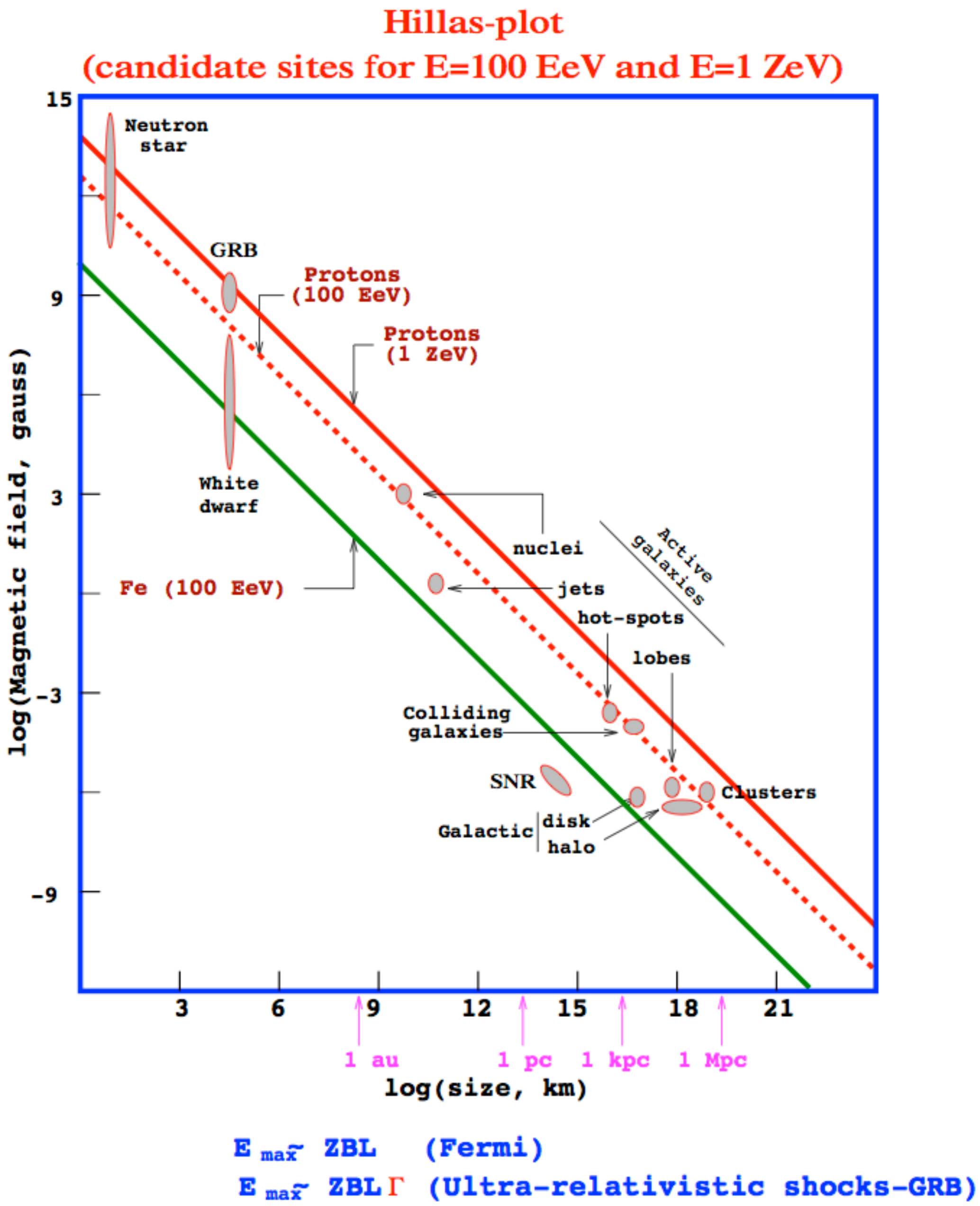

Fig. 3.10: Hillas diagram constructed by plotting different sources characteristics in the (size, B) plane. Sources above the green line may be able to accelerate Fe nuclei above 10^{20} eV, while those above the dashed and full red lines may be able to accelerate protons above 10^{20} and 10^{21} eV respectively. Note that the Hillas must be interpreted with the greatest caution : *(i)* The Hillas criterion is a necessary but not sufficiently condition to reach a given maximum energy. *(ii)* The maximum energies estimated using the Hillas criterion are overoptimistic (see text). *(iii)* Energy losses during the acceleration process (which may limit the maximum reachable energy) are not taken into account.

energy losses, which may limit the acceleration process (see next chapter) and which are specific to a given type of source, are not accounted for in this argument.

A slightly higher level argument can be used to derive a more quantitative estimate. In diffusion theory, one can show that the typical distance from the shock a particle manages to reach upstream or downstream and which is called the diffusion length is given by :

$$L_{diff} \simeq \frac{D(E)}{v_{sh}} \tag{3.55}$$

when the value of L_{diff} reaches a value of the order of the spatial extension of the magnetized region upstream of the shock then particle escape in the upstream region becomes efficient (and the approximation we made for our calculation is broken). We can then define a maximum energy due to particle confinement E_{max}^{conf} such that $L_{diff}(E_{max}^{conf}) = L_{up}$, where L_{up} is the size of the magnetized region upstream of the shock. Since particles with energy of the order of E_{max}^{conf} will escape efficiently upstream rather than re-crossing the shock, this escape mechanism is going to limit the maximum energy achievable by cosmic-rays which was not the case in our idealized calculation. This is however a good thing since the particles escaping downstream of the shock were not accelerated anymore but were still trapped within the source without any possibility to escape through the interstellar medium until the phenomenon at the origin of the acceleration dissipates. These particles would be likely to experience energy losses while being confined in the source. Particles escaping upstream by de-confinement eventually escape through the interstellar medium and can propagate throughout the Galaxy.

Particles with energies close to E_{max}^{conf} can escape efficiently upstream while at much lower energy the probability for reaching the boundary of the magnetized region upstream of the shock is very low[10]. For these low energy particles the above approximation of an infinite spatial extension for the upstream medium is relatively accurate.

$$L_{diff}(R) \simeq 110 \times \left(\frac{R}{10^{15}\mathrm{V}}\right) \times \left(\frac{B}{\mu\mathrm{G}}\right)^{-1} \times \left(\frac{v_{sh}}{1000\,\mathrm{km\,s^{-1}}}\right)^{-1} \mathrm{pc} \tag{3.56}$$

Using the same supernova remnant parameters as before and assuming $L_{up} = 0.1$ pc (the size of the upstream magnetised region is often assumed to be of the order of 1/10th of the size of remnant for which we use $L_{source} = 1$ pc). We get $E_{max}^{conf} \simeq Z \times 3\,10^{14}$ eV (or $R_{max}^{conf} \simeq 3\,10^{14}$ V) which is indeed much smaller than the above calculated E_{max}^{size}.

The important thing to understand here is that particles escaping upstream are directly released in the ISM and will eventually propagate through the Galaxy. They are the cosmic-rays we expect to detect on Earth. Since only particles with rigidities close to R_{max}^{conf} have a significant probability to escape upstream by reaching the boundary of the magnetized region, the spectrum of particles escaping upstream is expected to be much harder than the spectrum of accelerated cosmic-rays (one might call this phenomenon a "high pass filter effect"). This very important fact is illustrated in Fig. 3.11.

Another source of limitation for the acceleration of cosmic-rays might come from the "age" of the source (in other word the amount of time during which the source has been active). This criterion

[10]Since the escape upstream is a diffusive process the probability for particles with energies $E \ll E_{max}^{conf}$ is very small but never strictly 0.

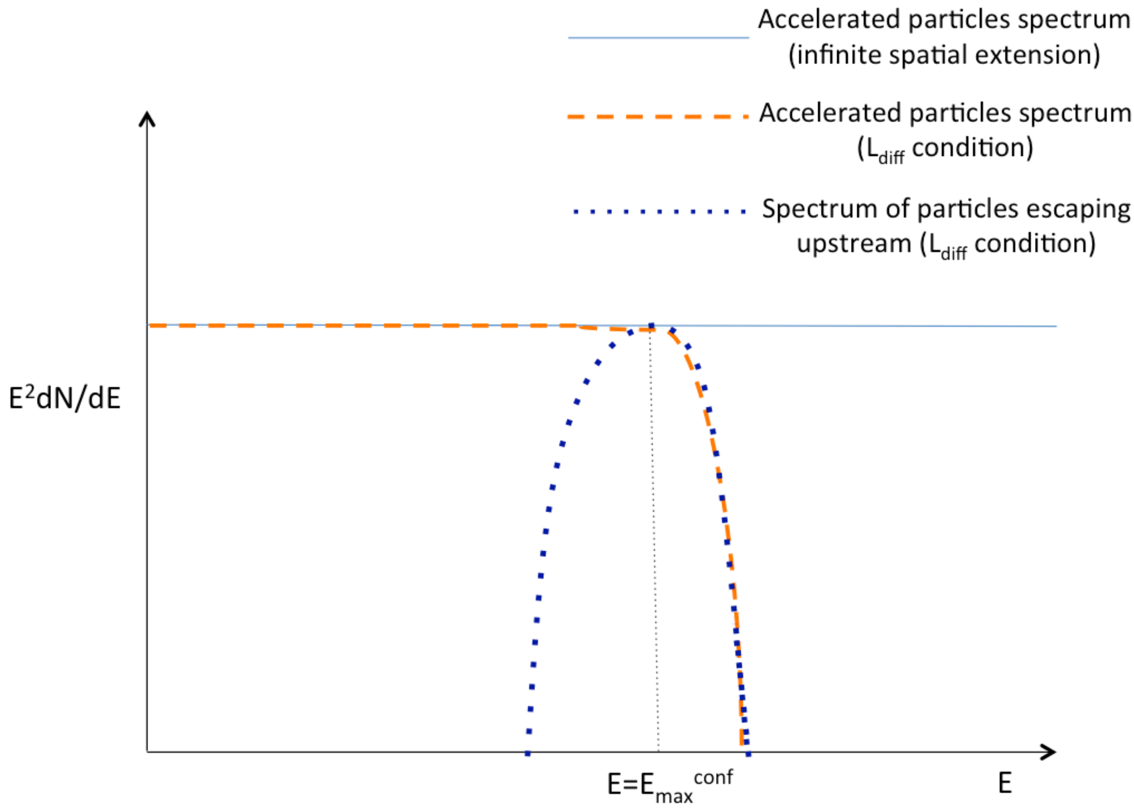

Fig. 3.11: Schematic view explaining the impact of the existence of the diffusion length and the limited confinement capabilities of a source : When the spatial extension of the source is assumed to be infinite, the spectrum of the accelerated cosmic-rays is proportional to E^{-2} and can reach arbitrarily large energies (blue line). When the spatial extension of the source is considered (and the condition on $L_{diff}(R)$ applied), the spectrum of the accelerated cosmic-rays is still proportional to E^{-2} but the acceleration mechanism becomes inefficient above E_{max}^{conf} since particles start to escape upstream of the shock rather than coming back to the shock and starting a new cycle (orange dashed line). Finally the dark blue dotted line shows the spectrum of particles which manage to escape upstream of the shock. The spectrum is very hard (almost a Dirac distribution) since the escape probability only becomes significant for energies close to E_{max}^{conf}. Above this energy the efficient escape upstream prevent the particles from performing extra cycles and reaching higher energies.

is relatively straightforward to understand : if a source has an age t_{source} then only the rigidities for with $t_{acc}(R) \leq t_{source}$ can be potentially reached.

For instance, we saw earlier that $\langle t_{acc}(10^{15}\ \mathrm{V})\rangle \simeq 480$ yr. Thus for an SNR with $B = 100\ \mu$G and $v_{sh} = 3000\ \mathrm{km\,s^{-1}}$ the maximum rigidity due to the age of the source after 480 yr will be $R_{max}^{age} \simeq 10^{15}$ V. Since in the Bohm hypothesis, $\langle t_{acc}(R) \propto R\rangle$ then if v_{sh} and B were constant with time, we would expect $R_{max}^{age} \propto t$.

Let us know discuss the comparison between R_{max}^{age} and R_{max}^{conf}, assuming for the time being v_{sh} and B are constant with time. After 480 yr, we have $L_{source} = v_{sh} \times t \simeq 1.45$ pc. Let us take $L_{up} \simeq 0.1\, L_{source} \simeq 0.145$ pc. Then we would get with the same calculation as above $R_{max}^{conf}(480\ \mathrm{yr}) \simeq 5\,10^{14}$ V. Then at $t = 480$ yr we have $R_{max}^{conf} \simeq R_{max}^{age}$ (within a factor of 2) and it should remain so at least while v_{sh} and B are constant (in which case as we saw $R_{max}^{age} \propto t$ and moreover $L_{up} \propto L_{source} \propto t \Rightarrow R_{max}^{conf} \propto t$.

In this situation the escape of particles becomes efficient at a rigidity which is practically the same as the maximum rigidity the cosmic accelerator can achieve (due to its age). This is actually a quite optimum situation and we can understand this points by considering two extreme case :

(i) if $R_{max}^{conf} \gg R_{max}^{age}$, in this case, the escape of particles at the maximum energy achievable is inefficient and then the accelerated particles remain trapped in the source.

(ii) if $R_{max}^{conf} \ll R_{max}^{age}$, in this case, particles escape becomes efficient at rigidities much lower than the maximum energy the source could achieve. This efficient escape would "break" the acceleration mechanism (see above) before reaching the rigidity R_{max}^{age}.

The case $R_{max}^{conf} \simeq R_{max}^{age}$ is then optimum since the criteria for an efficient particle escape upstream and for the acceleration of the particles to the maximum possible energy due to the age of the source are met at the same time.

In the next chapter, we will build a toy model of the time evolution of a supernova remnant. We will discuss in some more details how the different limitations for the maximum energy achievable in SNR compete with each other and how this competition changes with time. We will also study the influence of energy losses and the different constraints on cosmic-ray acceleration in SNRs that can be obtained using multi-wavelength observations.