Massless elementary particles with continuous spin

Wigner's exotic representation of the Poincaré group

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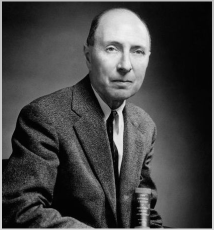
From joint works with

- **J. Mourad**, JHEP **0601** (2006) 115
  [hep-th/0509092]
- **M. Najafizadeh** and **M. R. Setare**, Phys. Lett. B **760** (2016) 320
  [arXiv:1506.00973 [hep-th]]
- **E. D. Skvortsov**, Int. J. Mod. Phys. A **32** (2017) 1730019
  [arXiv:1708.01030 [hep-th]].
- **J. Mourad** and **M. Najafizadeh**, JHEP **1711** (2017) 113
  [arXiv:1710.05788 [hep-th]]

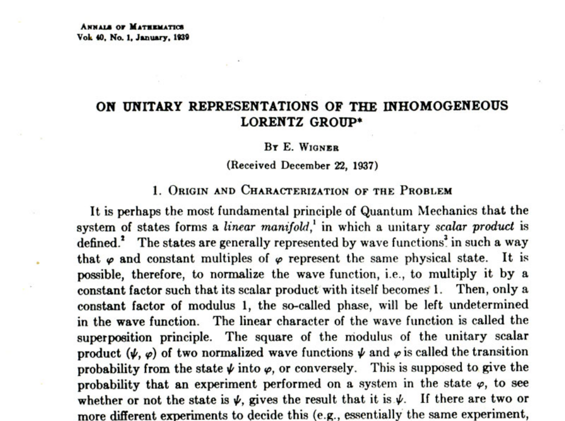
Wigner's exotic representation of the Poincaré group

Wigner taught us that free elementary particles propagating in flat spacetime are in one-to-one correspondence with unitary irreducible representations of the Poincaré group. This motivated his classification of all such representations in 1939.



Wigner's exotic representation of the Poincaré group

In Wigner's classification, the so-called "continuous spin" representations are actually the *generic* massless representations. However elementary particles described by continuous-spin representations are usually discarded on the basis of two exotic features.



ANNALS OF MATHEMATICS
Vol. 40, No. 1, January, 1939

ON UNITARY REPRESENTATIONS OF THE INHOMOGENEOUS LORENTZ GROUP*

BY E. WIGNER

(Received December 22, 1937)

1. ORIGIN AND CHARACTERIZATION OF THE PROBLEM

It is perhaps the most fundamental principle of Quantum Mechanics that the system of states forms a *linear manifold*,[1] in which a unitary *scalar product* is defined.[2] The states are generally represented by wave functions[3] in such a way that φ and constant multiples of φ represent the same physical state. It is possible, therefore, to normalize the wave function, i.e., to multiply it by a constant factor such that its scalar product with itself becomes 1. Then, only a constant factor of modulus 1, the so-called phase, will be left undetermined in the wave function. The linear character of the wave function is called the superposition principle. The square of the modulus of the unitary scalar product (ψ, φ) of two normalized wave functions ψ and φ is called the transition probability from the state ψ into φ, or conversely. This is supposed to give the probability that an experiment performed on a system in the state φ, to see whether or not the state is ψ, gives the result that it is ψ. If there are two or more different experiments to decide this (e.g., essentially the same experiment,

Wigner's exotic representation of the Poincaré group

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They are characterised by a continuous parameter with the dimension of a mass, although they are massless.

Second exotic property:
They have infinitely many degrees of freedom per spacetime point.

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- The meaning of the second property is that the helicities in their spectrum are unbounded, which is why Wigner later proposed the alternative terminology "infinite spin". More precisely, they are described by the (countable) infinite tower of all (either integer or half-integer) helicity states mixing under Lorentz boosts.

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- The infinite # of degrees of freedom (per spacetime point) was the main reason of Wigner's rejection of the continuous spin representation. He argued that this property implied that the heat capacity of a gas of such particles would be infinite.

Wigner's exotic representation of the Poincaré group

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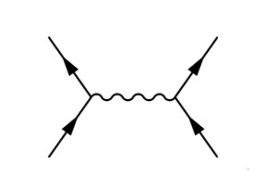
However, Schuster and Toro proposed in 2013 a class of soft factors for these massless particles, from which their phenomenology was argued to be much better behaved than naively expected. More precisely:

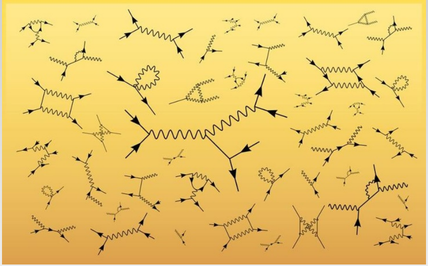
- These particles might circumvent Weinberg's no-go theorem on long-range interactions mediated by massless particles of spin higher than two in flat spacetime.
- At energies higher than the characteristic mass parameter alluded above, they may experience "helicity correspondence" in that they effectively behave like massless particles with helicities not higher than two.



Wigner's exotic representation of the Poincaré group

In order to investigate the properties of these exotic particles without any prejudice and reach conclusions on their phenomenological viability, one should first develop their field-theoretic description on a first-principle basis:

- **kinematics:**
  present covariant descriptions (linear eqs, quadratic Lagrangians)
- **dynamics:**
  classify their consistent interactions (vertices, scattering amplitudes)



(http://wwww.interest.com)

Outline

1. Group theory of the "continuous" spin representations
   - Generic massless unitary irreducible representations
   - Infinite-spin/Massless limit of massive representations
2. Bosonic and fermionic equations
   - Bosonic equations
   - Fermionic equations
3. Action principles
   - Bosonic action
   - Fermionic action
4. Summary of results and open problems
   - Some results
   - List of open problems

Group theory of the "continuous" spin representations

Wigner's classification: a reminder

Let us start by reviewing the representation theory of the Poincaré group. Actually, we will restrict our attention to the Lie algebra for simplicity.

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Unitary irreducible representations (UIRs) of the Poincaré algebra

- The Poincaré algebra $\mathfrak{iso}(D-1,1)=\mathbb{R}^D \ni\!\!\!\!+ \mathfrak{so}(D-1,1)$ is the semidirect sum of an Abelian and a semisimple Lie algebra

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UIR	Orbit	Stability	# of components
Massive	2-sheeted hyp $p^2 = -m^2$	$\mathfrak{so}(D-1)$	finite
Massless	light-cone $p^2 = 0$	$\mathfrak{iso}(D-2)$	finite or ∞
Tachyonic	1-sheeted hyp $p^2 = +m^2$	$\mathfrak{so}(D-2,1)$	1 or ∞
Zero-momentum	origin $p_\mu = 0$	$\mathfrak{so}(D-1,1)$	unfaithful irrep

Wigner's classification: massives representations

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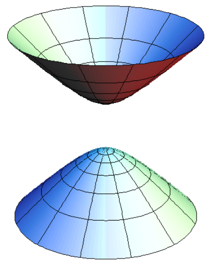

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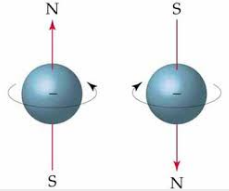
Unitary irreducible representations of the stability subalgebra

The rotation algebra $\mathfrak{so}(D-1)$ is compact and semisimple.

$\Rightarrow$ only finite-dimensional UIRs

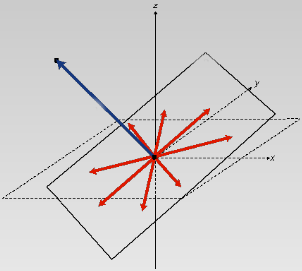
$\Rightarrow$ always finite # of components



Wigner's classification: massless representations

Consider a massless particle in D-dimensional spacetime with light-like momentum p_μ ($\mu = 0, 1, 2, \cdots, D-1$).

A spacelike plane orthogonal to this light-like momentum will be called a "transverse plane" $\mathbb{R}^{D-2} \subset \mathbb{R}^{D-1,1}$.



Wigner's classification: massless representations

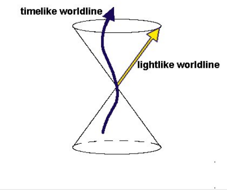
It is extremely convenient to use light-cone coordinates

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^{D-1})$$

adapted to the momentum, i.e. the latter has zero components except for $p^+ = -p_-$. The Minkowski metric reads

$$ds^2 = -2\,dx^+dx^- + dx^i dx_i\,, \qquad (i = 1, 2, \cdots, D-2)\,,$$

where x^i are Cartesian coordinates on the transverse plane.



Wigner's classification: massless representations

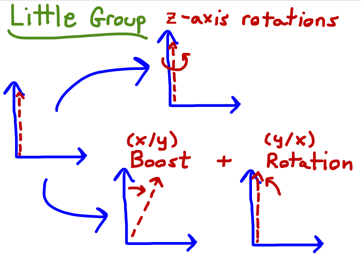
The massless little group $ISO(D-2)$ leaving the momentum invariant is formed of

- **rotations of the transverse plane**, generated by

$$\hat{M}_{ij} \qquad (i,j = 1,2,\cdots,D-2)$$

- **transverse null boosts**, generated by

$$\hat{\pi}_i := \hat{M}_{+i} = \frac{1}{\sqrt{2}}\big(\hat{M}_{0i} + \hat{M}_{D-1\,i}\big) \qquad (i = 1,2,\cdots,D-2)\,.$$



(P. Schuster and N. Toro, arXiv:1302.1198 [hep-th])

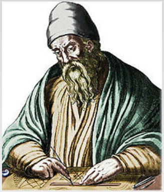
Wigner's classification: massless representations

The generators span a Lie algebra isomorphic to the Euclidean algebra $\mathfrak{iso}(D-2)$ of the transverse plane

$$\left[\hat{M}_{ij}, \hat{M}_{kl}\right] = i\left(\delta_{jk}\hat{M}_{il} - \delta_{ik}\hat{M}_{jl} - \delta_{jl}\hat{M}_{ik} + \delta_{il}\hat{M}_{jk}\right),$$

$$\left[\hat{\pi}_i, \hat{M}_{kl}\right] = i\left(\delta_{ik}\hat{\pi}_l - \delta_{il}\hat{\pi}_k\right),$$

$$[\hat{\pi}_i, \hat{\pi}_j] = 0\,.$$



Wigner's classification: massless representations

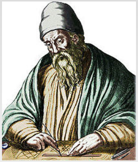
UIR	Orbit	Stability	# of components
Massless	light-cone $p^2 = 0$	$\mathfrak{iso}(D-2)$	finite (helicity) vs ∞

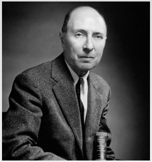
Unitary irreducible representations of the stability subalgebras

The Euclidean algebra $\mathfrak{iso}(D-2)$ of the transverse plane is noncompact.

$\Rightarrow$ only infinite-dimensional *faithful* UIRs

$\Rightarrow$ either finite # of components but unfaithful of $\mathfrak{iso}(D-2)$,
or infinite # of components and faithful of $\mathfrak{iso}(D-2)$.





Wigner's classification: massless representations

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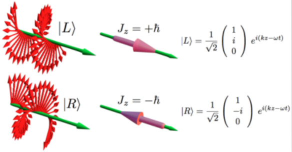
- **Helicity (or "finite-spin"):**
  Trivial representation of the transverse "translation" subalgebra $\mathbb{R}^{D-2}\subset\mathfrak{iso}(D-2)$.
  Effectively, UIR of the transverse rotation subalgebra $\mathfrak{so}(D-2)\subset\mathfrak{iso}(D-2)\Rightarrow$ finite-dimensional UIR

  $\Rightarrow$ finite # of components ("finite spin")



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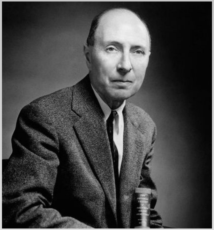
- **Continuous spin (or "infinite-spin"):**
  Nontrivial representation of the transverse "translation" subalgebra
  $\mathbb{R}^{D-2} \subset \mathfrak{iso}(D-2) \Rightarrow$ infinite-dimensional UIR

  $\Rightarrow$ infinite # of components ("infinite spin")

Wigner's classification: Euclidean group

Let us now consider more closely the representation of the little group by applying the method of induced representations to the Euclidean group. (Actually, we will again restrict our attention to the Lie algebra.)



Wigner's classification: Euclidean group

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- The Euclidean algebra $\mathfrak{iso}(d) = \mathbb{R}^d \niplus \mathfrak{so}(d)$ is the semidirect sum of an Abelian and a semisimple Lie algebra

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  1. Consider the UIRs of the Abelian subalgebra: labelled by real eigenvalues (unitary & irreducible) of generators $\hat{\pi}_i$
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UIR	Dimension	Orbit	Stability	Example
Faithful	infinite	sphere $\eta^2 = \mu^2$	$\mathfrak{so}(d-1)$	Solutions of Helmholtz eq
Unfaithful	finite	origin $\eta_i = 0$	$\mathfrak{so}(d)$	Spherical harmonics

Wigner's classification: massless representations

In terms of massless representations, the former classification of the UIR of their little group provides the distinction between "helicity" (finite-components) and "continuous-spin" (infinite-components) representations.

Wigner's classification: massless representations

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UIR	# of components	Orbit	Stability
Continuous spin	infinite	**sphere** $\eta^2 = \mu^2$	$\mathfrak{so}(D-3)$
Helicity	finite	**origin** $\eta_i = 0$	$\mathfrak{so}(D-2)$

Wigner's classification: helicity representations

UIR	# of components	Orbit	Stability
Helicity	finite	origin $\eta_i = 0$	$\mathfrak{so}(D-2)$

Remarks:

- The subgroup $SO(D-2)$ can be called the "effective little group" of helicity representations. It admits nontrivial representations for spacetime dimensions $D \geqslant 5$ only.

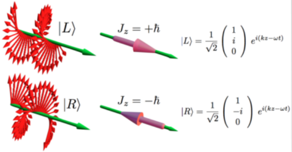
Wigner's classification: helicity representations

UIR	# of components	Orbit	Stability
Helicity	finite	**origin** $\eta_i = 0$	$\mathfrak{so}(D-2)$

Remarks:

- The subgroup $SO(D-2)$ can be called the "effective little group" of helicity representations. It admits nontrivial representations for spacetime dimensions $D \geqslant 5$ only.

  $\Rightarrow$ In four dimensions, finite-spin representations are labelled by a single real number: their helicity.



Wigner's classification: continuous-spin representations

UIR	# of components	Orbit	Stability
Continuous spin	infinite	(D-3)-sphere $\eta^2 = \mu^2$	$\mathfrak{so}(D-3)$

Remarks:

- The stabiliser group $SO(D-3)$ of the transverse vector η_i is sometimes called the "short little group" of continuous-spin representations. This group is degenerate for $D \leqslant 3$ and admits nontrivial representations for $D \geqslant 6$ only.

Wigner's classification: continuous-spin representations

UIR	# of components	Orbit	Stability
Continuous spin	infinite	(D-3)-sphere $\eta^2 = \mu^2$	$\mathfrak{so}(D-3)$

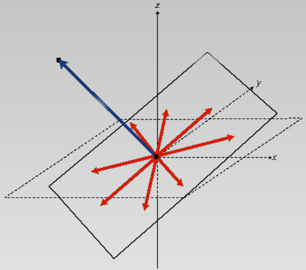
Remarks:

- The stabiliser group $SO(D-3)$ of the transverse vector η_i is sometimes called the "short little group" of continuous-spin representations. This group is degenerate for $D \leqslant 3$ and admits nontrivial representations for $D \geqslant 6$ only.

  $\Rightarrow$ In four dimensions, $\exists$ only two infinite-spin representations: the single-valued (bosonic) and the double-valued (fermionic) ones whose physical components span all (integer or half-integer) helicities.

Wigner's classification: continuous-spin representations

For $D = 4$ spacetime dimension, the physical components forming an UIR of the little group $ISO(2)$ can be realised as square-integrable functions on the circle in the transverse plane.



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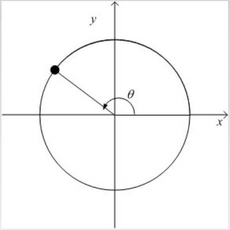
- "**angle basis**": The states $\mid\theta\,\rangle$ are eigenstates of the null boosts ("translations") but transverse rotations transform these states into each other $\mid\theta\,\rangle \rightarrow \mid\theta+\alpha\,\rangle$.



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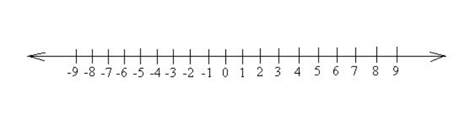

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- "**helicity basis**": The Fourier dual basis elements $\mid h \,\rangle$ are eigenstates of the transverse rotation generator, but the null boosts mix this infinite tower of helicity eigenstates.

Remark: The mixing disappears in the limit $\mu \rightarrow 0$ for which the continuous-spin representation becomes the direct sum of all (either integer or half-integer) helicity representations. In more physical terms: at energies $E \gg \mu$, free continuous-spin particles behave as an infinite tower of particles with distinct helicities.

Casimir operators of the Poincaré algebra

Another traditional method for classifying UIRs of the Poincaré group is to make use of the eigenvalues of the Casimir operators.

+ In this approach, Lorentz covariance is more direct and the physical interpretation of Casimir operators (square of momentum and Pauli-Lubanski vectors in $D = 4$) may be more enlightening than in the method of induced representations.

Casimir operators of the Poincaré algebra

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+ In this approach, Lorentz covariance is more direct and the physical interpretation of Casimir operators (square of momentum and Pauli-Lubanski vectors in $D = 4$) may be more enlightening than in the method of induced representations.
- The UIRs of (finite-dimensional) semisimple Lie algebras are characterised uniquely by the eigenvalues of their independent Casimir operators. However, this is not necessarily true for non-semisimple Lie algebras (such as Poincaré algebra), for which there can be degeneracies (e.g. all helicity representations have vanishing quadratic and quartic Casimir operators).

Casimir operators of the Poincaré algebra

The quadratic Casimir operator of the Lorentz algebra $\mathfrak{so}(D-1,1)$ is the square of the generators $M_{\mu\nu}$:

$$\hat{\mathcal{C}}_2\Big(\mathfrak{so}(D-1,1)\Big) \;=\; \frac{1}{2}\,\hat{M}^{\mu\nu}\hat{M}_{\mu\nu}\,.$$

The quadratic Casimir operator of the Poincaré algebra $\mathfrak{iso}(D-1,1)$ is the square of the momentum

$$\hat{\mathcal{C}}_2\Big(\mathfrak{iso}(D-1,1)\Big) \;=\; -\hat{P}^{\mu}\hat{P}_{\mu}\,,$$

while the quartic Casimir operator is

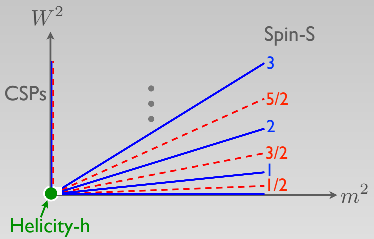
$$\hat{\mathcal{C}}_4\Big(\mathfrak{iso}(D-1,1)\Big) \;=\; -\frac{1}{2}\hat{P}^2\hat{M}_{\mu\nu}\hat{M}^{\mu\nu}+\hat{M}_{\mu\rho}\hat{P}^{\rho}\hat{M}^{\mu\sigma}\hat{P}_{\sigma}\,,$$

which, for $D=4$, is the square of the Pauli-Lubanski vector,

$$\hat{W}^{\mu} := \frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\hat{M}_{\nu\rho}\hat{P}_{\sigma}\,.$$

Infinite-spin/Massless limit of massive representations

$$\mathcal{C}_4\Big(\mathfrak{iso}(3,1)\Big) = W^2 = \begin{cases} m^2\, s(s+1) & (m^2 \neq 0) \\ \mu^2 & (m^2 = 0) \end{cases}$$



(P. Schuster and N. Toro, `arXiv:1302.1198 [hep-th]`)

Infinite-spin/Massless limit of massive representations

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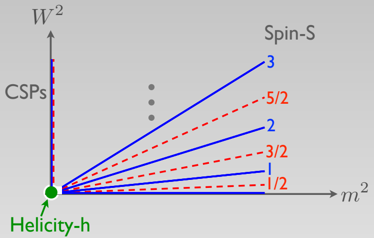

Idea: (Khan & Ramond, 2005) Obtain the continuous-spin representation from the massive representation in the limit

$$m \to 0\,, \qquad s \to \infty\,, \qquad \mu = m\,s \quad \text{fixed}$$

Infinite-spin/Massless limit of massive representations

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This perspective on the continuous-spin provides a simple and physical explanation of both exotic properties.

First exotic property:
They are characterised by a continuous parameter with the dimension of a mass, although they are massless.

Second exotic property:
They have infinitely many degrees of freedom per spacetime point.

Infinite-spin/Massless limit of massive representations

Idea: (Khan & Ramond, 2005) Obtain the continuous-spin representation from the massive representation in the limit

$$m \to 0\,, \qquad s \to \infty\,, \qquad \mu = m\, s \quad \text{fixed}$$

At the level of the little group, this limit is related to the Inönü-Wigner contraction

$$\mathfrak{so}(D-1) \stackrel{m\to 0}{\longrightarrow} \mathfrak{iso}(D-2)\,.$$

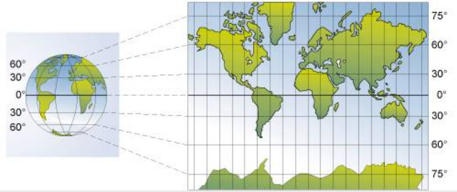
Infinite-spin/Massless limit of massive representations

Geometrical interpretation: The Inönü-Wigner contraction of isometry algebras

$$\mathfrak{so}(d) \overset{R\to\infty}{\longrightarrow} \mathfrak{iso}(d-1)$$

corresponds geometrically to the flat limit of the sphere,

$$\mathbb{S}^{d-1} \overset{R\to\infty}{\longrightarrow} \mathbb{R}^{d-1}\,.$$



Infinite-spin/Massless limit of massive representations

Simplest example: The simplest UIR of the rotation algebra $\mathfrak{so}(d)$ are the spherical harmonics, i.e. the solutions of the equation

$$\left(\Delta_{\mathbb{S}^{d-1}} + \frac{s(s+d-2)}{R^2}\right) Y^s_m(\vec{\theta}) = 0$$

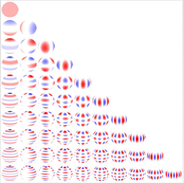
whose limit

$$R \to \infty\,, \qquad s \to \infty\,, \qquad \mu = s/R \quad \text{fixed}$$

is the Helmholtz equation

$$\left(\Delta_{\mathbb{R}^{d-1}} + \mu^2\right) \Phi(\vec{x}) = 0\,,$$

whose space of solutions carries the simplest UIR of the Euclidean algebra $\mathfrak{iso}(d-1)$.



Bosonic and Fermionic Equations

Infinite-spin/Massless limit of massive equations

- In the 1930's and 1940's, various equivalent covariant equations have been proposed, the solution space of which carries a massive UIR of the Poincaré group.
- In 1947, Wigner proposed covariant equations, the solution space of which carries an infinite-spin massless UIR of the Poincaré group.

Infinite-spin/Massless limit of massive equations

- In the 1930's and 1940's, various equivalent covariant equations have been proposed, the solution space of which carries a massive UIR of the Poincaré group.
- In 1947, Wigner proposed covariant equations, the solution space of which carries an infinite-spin massless UIR of the Poincaré group.

Retrospectively, Wigner equations can be obtained from a suitable infinite-spin massless limit of the massive equations (XB & Mourad, 2005).

Bosonic equations

Higher-spin massive equations

Higher-spin massive equations (Dirac, Fierz, Pauli, ...)

For integer spin $s \in \mathbb{N}$, covariant equations carrying the massive UIR of the Poincaré group $ISO(D-1,1)$ induced from the symmetric tensor representation of the little group $SO(D-1)$ can be formulated in terms of a symmetric Lorentz tensor:

$$(p^2 + m^2)\varphi_{\mu_1\mu_2\ldots\mu_s}(p) = 0$$
$$p^\nu \varphi_{\nu\mu_1\mu_2\ldots\mu_{s-1}}(p) = 0$$
$$\varphi^\nu{}_{\nu\mu_1\mu_2\ldots\mu_{s-2}}(p) = 0$$

Indeed, in a rest frame the second equation implies the vanishing of all timelike components while the third equation implies the $\mathfrak{so}(D-1)$-irreducibility.

Higher-spin massive equations

Standard trick for higher-spins: Contract all indices with an auxilliary vector, say u^μ, and introduce the generating function

$$\varphi(p,u)=\frac{1}{s!}\varphi_{\mu_1\dots\mu_s}(p)\,u^{\mu_1}\cdots u^{\mu_s}$$

so that the massive equations read

$$\begin{aligned}
(p^2+m^2)\varphi_{\mu_1\dots\mu_s}(p)=0 \quad &\Longleftrightarrow \quad (p^2+m^2)\varphi(p,u)=0\\
p^\nu\varphi_{\nu\mu_1\dots\mu_{s-1}}(p)=0 \quad &\Longleftrightarrow \quad \left(p\cdot\frac{\partial}{\partial u}\right)\varphi(p,u)=0\\
\varphi^\nu{}_{\nu\mu_1\dots\mu_{s-2}}(p)=0 \quad &\Longleftrightarrow \quad \left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}\right)\varphi(p,u)=0
\end{aligned}$$

To which, we should now add the homogeneity equation to keep track of the spin:

$$\left(u\cdot\frac{\partial}{\partial u}-s\right)\varphi(p,u)=0$$

Infinite-spin/Massless limit of massive equations

Unfortunately, this homogeneity equation is singular in the limit $s \to \infty$.

Infinite-spin/Massless limit of massive equations

Unfortunately, this homogeneity equation is singular in the limit $s \to \infty$.

Nevertheless, the massless limit with fixed spin is well defined.

Finite-spin/Massless limit of massive equations

Finite-spin massless equations

The massless limit $m \to 0$ of the massive equations (with spin s fixed) are

$$\begin{aligned} p^2\,\varphi(p,u) &= 0 \\ \left(p\cdot\frac{\partial}{\partial u}\right)\varphi(p,u) &= 0 \\ \left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}\right)\varphi(p,u) &= 0 \\ \left(u\cdot\frac{\partial}{\partial u}-s\right)\varphi(p,u) &= 0 \end{aligned}$$

Finite-spin/Massless limit of massive equations

Finite-spin massless equations

The massless limit $m \to 0$ of the massive equations (with spin s fixed) are

$$p^2\,\varphi(p,u) = 0$$
$$\left(p\cdot\frac{\partial}{\partial u}\right)\varphi(p,u) = 0$$
$$\left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}\right)\varphi(p,u) = 0$$
$$\left(u\cdot\frac{\partial}{\partial u} - s\right)\varphi(p,u) = 0$$

However, these equations propagate too many degrees of freedom.

Finite-spin/Massless limit of massive equations

In order for the space of solutions to carry the helicity UIR of the Poincaré group $ISO(D-1,1)$ induced from the symmetric tensor representation of the effective little group $SO(D-2)$, gauge equivalent solutions must be identified:

$$\varphi(p,u) \sim \varphi(p,u) + (u \cdot p)\, \varepsilon(p,u)\,.$$

In other words, longitudinal components are pure gauge:

$$\varphi_{\mu_1 \ldots \mu_s}(p) \sim \varphi_{\mu_1 \ldots \mu_s}(p) + p_{(\mu_1} \varepsilon_{\mu_2 \ldots \mu_s)}(p)\,.$$

Massless equations

Strictly speaking, for consistency the gauge parameter $\varepsilon(p,u)$ should obey to similar equations

$$p^2\,\varepsilon(p,u)=0$$

$$\left(p\cdot\frac{\partial}{\partial u}\right)\varepsilon(p,u)=0$$

$$\left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}\right)\varepsilon(p,u)=0$$

$$\left(u\cdot\frac{\partial}{\partial u}\;-\;(s-1)\right)\varepsilon(p,u)=0$$

Massless equations

One way to get rid of the equivalence relation is to leave the space of polynomials in the auxilliary vector and reformulate the equations in terms of the gauge-invariant distribution

$$\phi(p,u) \;=\; \delta(p\cdot u)\,\varphi(p,u)\,.$$

Finite-spin massless equations

$$p^2\,\phi(p,u) = 0$$

$$(p\cdot u)\,\phi(p,u) = 0,$$

$$\left(p\cdot\frac{\partial}{\partial u}\right)\phi(p,u) = 0$$

$$\left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}\right)\phi(p,u) = 0$$

$$\left(u\cdot\frac{\partial}{\partial u} - (s-1)\right)\phi(p,u) = 0$$

Dimensional reduction of massless equations

Remark: The effective little group of massless particles in $D+1$ dimensions is

$$SO\big((D+1)-2\big) = SO(D-1)$$

which coincides with the little group of massive particles in D dimensions.

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This is the group-theoretical explanation behind the technique of dimensional reduction for obtaining massive equations from massless equations in one higher dimension, by considering a single Kaluza-Klein mode.

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If the higher-dimensional massless fields are gauge fields, then the dimensional reduction typically produces a tower of lower-spin Stuckelberg (i.e. pure gauge) or auxilliary fields.

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If the higher-dimensional massless fields are gauge fields, then the dimensional reduction typically produces a tower of lower-spin Stuckelberg (i.e. pure gauge) or auxilliary fields.

Massive equations in such a Stuckelberg approach turn out to be more convenient for taking the infinite-spin massless limit of massive equations.

Dimensional reduction of massless equations

Finite-spin massless equations in D dimensions

$$
\begin{aligned}
& p^2\,\phi(p,u)=0\\
& (p\cdot u)\,\phi(p,u)=0,\\
& \left(p\cdot\frac{\partial}{\partial u}\right)\phi(p,u)=0\\
& \left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}\right)\phi(p,u)=0\\
& \left(u\cdot\frac{\partial}{\partial u}-(s-1)\right)\phi(p,u)=0
\end{aligned}
$$

Dimensional reduction of massless equations

Finite-spin massless equations in D+1 dimensions

$$P^2\,\Phi(P,U)=0$$
$$\big(P\cdot U\big)\,\Phi(P,U)=0,$$
$$\left(P\cdot\frac{\partial}{\partial U}\right)\Phi(P,U)=0$$
$$\left(\frac{\partial}{\partial U}\cdot\frac{\partial}{\partial U}\right)\Phi(P,U)=0$$
$$\left(U\cdot\frac{\partial}{\partial U}-(s-1)\right)\Phi(P,U)=0$$

Dimensional reduction of massless equations

Finite-spin massless equations in D+1 dimensions

$$P^2\,\Phi(P,U)=0$$
$$\big(P\cdot U\big)\,\Phi(P,U)=0,$$
$$\left(P\cdot\frac{\partial}{\partial U}\right)\Phi(P,U)=0$$
$$\left(\frac{\partial}{\partial U}\cdot\frac{\partial}{\partial U}\right)\Phi(P,U)=0$$
$$\left(U\cdot\frac{\partial}{\partial U}-(s-1)\right)\Phi(P,U)=0$$

Consider the splittings

$$P_M=(p_\mu,m)\,,\qquad U^M=(u^\mu,v)\,.$$

Dimensional reduction of massless equations

Finite-spin massive equations in D dimensions

$$
\begin{aligned}
&(p^2+m^2)\,\Phi(p,u,v)=0\\
&(p\cdot u+mv)\,\Phi(p,u,v)=0\\
&\left(p\cdot\frac{\partial}{\partial u}+m\frac{\partial}{\partial v}\right)\Phi(p,u,v)=0\\
&\left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}+\frac{\partial^2}{\partial v^2}\right)\Phi(p,u,v)=0\\
&\left(u\cdot\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}-(s-1)\right)\Phi(p,u,v)=0
\end{aligned}
$$

These massive equations are somehow a "gauge-fixed" version of the Stuckelberg formulation.

Infinite-spin/Massless limit of massive equations

Finite-spin massive equations in D dimensions

$$(p^2 + m^2)\, \Phi(p,u,v) = 0$$
$$(p \cdot u + mv)\, \Phi(p,u,v) = 0$$
$$\left(p \cdot \frac{\partial}{\partial u} + m\frac{\partial}{\partial v}\right) \Phi(p,u,v) = 0$$
$$\left(\frac{\partial}{\partial u} \cdot \frac{\partial}{\partial u} + \frac{\partial^2}{\partial v^2}\right) \Phi(p,u,v) = 0$$
$$\left(u \cdot \frac{\partial}{\partial u} + v\frac{\partial}{\partial v} - (s-1)\right) \Phi(p,u,v) = 0$$

Problem: It is clear from the last equation that the infinite-spin limit is ill defined in terms of the field Φ.

Infinite-spin/Massless limit of massive equations

Finite-spin massive equations in D dimensions

$$
\begin{aligned}
&(p^2+m^2)\,\Phi(p,u,v)=0\\
&(p\cdot u+mv)\,\Phi(p,u,v)=0\\
&\left(p\cdot\frac{\partial}{\partial u}+m\frac{\partial}{\partial v}\right)\Phi(p,u,v)=0\\
&\left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}+\frac{\partial^2}{\partial v^2}\right)\Phi(p,u,v)=0\\
&\left(u\cdot\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}-(s-1)\right)\Phi(p,u,v)=0
\end{aligned}
$$

Idea: In order to get a well defined limit, one has to extract an infinite factor from Φ and also to assume a suitable scaling of the variable v.

Infinite-spin/Massless limit of massive equations

Finite-spin massive equations in D dimensions

$$(p^2 + m^2)\,\Phi(p,u,v) = 0$$
$$(p\cdot u + mv)\,\Phi(p,u,v) = 0$$
$$\left(p\cdot\frac{\partial}{\partial u} + m\frac{\partial}{\partial v}\right)\Phi(p,u,v) = 0$$
$$\left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u} + \frac{\partial^2}{\partial v^2}\right)\Phi(p,u,v) = 0$$
$$\left(u\cdot\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} - (s-1)\right)\Phi(p,u,v) = 0$$

Let us introduce the parameter μ and the variable α by

$$\mu = s\,m, \quad \alpha = v/s\,.$$

The precise limit we are interested in, is when s goes to infinity, with finite μ and α.

Infinite-spin/Massless limit of massive equations

Consider the limit

$$m \to 0\,, \qquad s \to \infty\,, \qquad \mu = m\,s \quad \text{fixed}$$

and change of auxilliary variables (u^μ, v) to the new variables (ω^μ, α)

$$\begin{cases} u^\mu = \omega^\mu\,\alpha \\ v = s\,\alpha \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \omega^\mu = \dfrac{s}{v}\,u^\mu \\ \alpha = \dfrac{v}{s} \end{cases}$$

that will be kept finite.

Infinite-spin/Massless limit of massive equations

Consider the limit

$$m \to 0\,, \qquad s \to \infty\,, \qquad \mu = m\,s \quad \text{fixed}$$

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$$\begin{cases} u^\mu = \omega^\mu\,\alpha \\ v = s\,\alpha \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \omega^\mu = \dfrac{s}{v}\,u^\mu \\ \alpha = \dfrac{v}{s} \end{cases}$$

that will be kept finite.

In fact, the problematic homogeneity condition can be solved as

$$\left(u \cdot \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} - (s-1) \right) \Phi = 0 \quad \Longleftrightarrow \quad \Phi = \alpha^{s-1}\Psi\left(\frac{u}{\alpha}\right)$$

and the remaining equations can all be expressed in terms of the field $\Psi(\omega) = \Psi(u/\alpha)$ which remains finite in the limit.

Infinite-spin/Massless limit of massive equations

Consider the limit

$$m \to 0\,, \qquad s \to \infty\,, \qquad \mu = m\,s \quad \text{fixed}$$

and change of auxilliary variables (u^μ, v) to the new variables (ω^μ, α)

$$\begin{cases} u^\mu = \omega^\mu\,\alpha \\ v = s\,\alpha \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \omega^\mu = \dfrac{s}{v}\,u^\mu \\ \alpha = \dfrac{v}{s} \end{cases}$$

that will be kept finite.

For instance, the second equation becomes

$$(p \cdot u + mv)\,\Phi = 0 \quad \Longleftrightarrow \quad (p \cdot \omega + \mu)\Psi = 0\,,$$

which actually motivated the change of variables.

Infinite-spin/Massless limit of massive equations

Finite-spin massive equations

$$\left[p^2+\left(\frac{\mu}{s}\right)^2\right]\Psi(p,\omega)=0$$

$$(p\cdot\omega+\mu)\,\Psi(p,\omega)=0$$

$$\left[p\cdot\frac{\partial}{\partial\omega}+\frac{\mu}{s^2}\left(s-1-\omega\cdot\frac{\partial}{\partial\omega}\right)\right]\Psi(p,\omega)=0$$

$$\left[\frac{\partial}{\partial\omega}\cdot\frac{\partial}{\partial\omega}+\frac{1}{s^2}\left((s-1)(s-2)-(2s-3)\left(\omega\cdot\frac{\partial}{\partial\omega}\right)+\left(\omega\cdot\frac{\partial}{\partial\omega}\right)^2\right)\right]\Psi=0$$

Infinite-spin/Massless limit of massive equations

Infinite-spin massless equations

$$p^2\,\Psi(p,\omega)=0$$

$$(p\cdot\omega+\mu)\,\Psi(p,\omega)=0$$

$$\left(p\cdot\frac{\partial}{\partial\omega}\right)\Psi(p,\omega)=0$$

$$\left(\frac{\partial}{\partial\omega}\cdot\frac{\partial}{\partial\omega}+1\right)\Psi(p,\omega)=0$$

Infinite-spin/Massless limit of massive equations

Infinite-spin massless equations

$$
\begin{aligned}
&p^2\,\Psi(p,\omega)=0\\
&(p\cdot\omega+\mu)\,\Psi(p,\omega)=0\\
&\left(p\cdot\frac{\partial}{\partial\omega}\right)\Psi(p,\omega)=0\\
&\left(\frac{\partial}{\partial\omega}\cdot\frac{\partial}{\partial\omega}+1\right)\Psi(p,\omega)=0
\end{aligned}
$$

Performing a Fourier transform over the auxilliary vector ω leads exactly to Wigner's equations in terms of the wave function

$$
\widetilde{\Psi}(p,\eta)=\int d\omega\;\Psi(p,\omega)\,\exp\big(-\,i\,(\eta\cdot\omega)/\mu\big)
$$

Wigner equations

Infinite-spin massless equations (Wigner, 1947)

$$
\begin{aligned}
&p^2\, \widetilde{\Psi}(p,\eta) = 0 \\
&(p\cdot\eta)\, \widetilde{\Psi}(p,\eta) = 0 \\
&\left(p\cdot\frac{\partial}{\partial\eta} - i\right) \widetilde{\Psi}(p,\eta) = 0 \\
&(\eta^2 - \mu^2)\, \widetilde{\Psi}(p,\eta) = 0
\end{aligned}
$$

Wigner equations

Infinite-spin massless equations (Wigner, 1947)

$$p^2\,\widetilde{\Psi}(p,\eta)=0$$
$$(p\cdot\eta)\,\widetilde{\Psi}(p,\eta)=0$$
$$\left(p\cdot\frac{\partial}{\partial\eta}-i\right)\widetilde{\Psi}(p,\eta)=0$$
$$(\eta^2-\mu^2)\widetilde{\Psi}(p,\eta)=0$$

Analysis: The physical components carry an UIR of the massless little group $ISO(D-2)$ because the auxilliary vector η belongs to the sphere $\mathbb{S}^{D-3}\subset\mathbb{R}^{D-2}$ inside the transverse plane.

Wigner equations

Infinite-spin massless equations (Wigner, 1947)

$$
\begin{aligned}
&p^2\,\widetilde{\Psi}(p,\eta)=0\\
&(p\cdot\eta)\,\widetilde{\Psi}(p,\eta)=0\\
&\left(p\cdot\frac{\partial}{\partial\eta}-i\right)\widetilde{\Psi}(p,\eta)=0\\
&(\eta^2-\mu^2)\,\widetilde{\Psi}(p,\eta)=0
\end{aligned}
$$

Proof:

- The 2nd equation implies that the support of the function is such that $\eta\perp p$.
- The 3rd equation is solved as

$$\widetilde{\Psi}(p\,,\eta+\phi\,p)\;=\;e^{i\,\phi}\,\widetilde{\Psi}(p\,,\eta)\,,\qquad\forall\theta\in\mathbb{R}$$

which shows that the longitudinal part of η is pure gauge.

Wigner equations

Infinite-spin massless equations (Wigner, 1947)

$$
\begin{aligned}
&p^2\,\widetilde{\Psi}(p,\eta)=0\\
&(p\cdot\eta)\,\widetilde{\Psi}(p,\eta)=0\\
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&(\eta^2-\mu^2)\,\widetilde{\Psi}(p,\eta)=0
\end{aligned}
$$

Proof:

- The 2nd and 3rd equations imply that one can assume that the auxilliary vector belongs to the transverse plane: $\eta\in\mathbb{R}^{D-2}$.

Wigner equations

Infinite-spin massless equations (Wigner, 1947)

$$\begin{aligned} p^2\,\widetilde{\Psi}(p,\eta) &= 0 \\ (p\cdot\eta)\,\widetilde{\Psi}(p,\eta) &= 0 \\ \left(p\cdot\frac{\partial}{\partial\eta} - i\right)\widetilde{\Psi}(p,\eta) &= 0 \\ (\eta^2-\mu^2)\,\widetilde{\Psi}(p,\eta) &= 0 \end{aligned}$$

Proof:

- The 2nd and 3rd equations imply that one can assume that the auxilliary vector belongs to the transverse plane: $\eta\in\mathbb{R}^{D-2}$.
- The 4th equation leads to the conclusion: $\eta\in\mathbb{S}^{D-3}$.

Fermionic equations

Higher-spin massive equations

Higher-spin massive equations (Dirac, Fierz, Pauli, ...)
For half-integer spin $s \in \mathbb{N} + \frac{1}{2}$, covariant equations carrying the doubled-valued massive UIR of the Poincaré group $ISO(D-1,1)$ induced from the symmetric spinor-tensor representation of the little group $Spin(D-1)$ can be formulated in terms of a symmetric Lorentz spinor-tensor (the spinor indices will be left implicit):

$$(\gamma^\mu p_\mu + m)\, \varphi_{\mu_1 \mu_2 \dots \mu_s}(p) = 0$$
$$p^\nu \varphi_{\nu \mu_1 \mu_2 \dots \mu_{s-1}}(p) = 0$$
$$\gamma^\nu \varphi_{\nu \mu_1 \mu_2 \dots \mu_{s-1}}(p) = 0$$

Indeed, in a rest frame the second equation implies the vanishing of all timelike components while the third equation implies the $\mathfrak{so}(D-1)$-irreducibility.

Higher-spin massive equations

In terms of the generating function

$$\varphi(p,u) = \frac{1}{s!}\,\varphi_{\mu_1\mu_2\ldots\mu_s}(p)\,u^{\mu_1}\cdots u^{\mu_s}$$

Higher-spin massive equations (Dirac, Fierz, Pauli, ...)

$$(\gamma\cdot p + m)\,\varphi(p,u) = 0$$

$$\left(p\cdot\frac{\partial}{\partial u}\right)\varphi(p,u) = 0$$

$$\left(\gamma\cdot\frac{\partial}{\partial u}\right)\varphi(p,u) = 0$$

$$\left(u\cdot\frac{\partial}{\partial u} - s\right)\varphi(p,u) = 0$$

Finite-spin/Massless limit of massive equations

Higher-spin massless equations

The massless limit $m \to 0$ of the massive equations (with spin s fixed) is

$$(\gamma \cdot p)\, \varphi(p,u) = 0$$
$$\left(p \cdot \frac{\partial}{\partial u}\right) \varphi(p,u) = 0$$
$$\left(\gamma \cdot \frac{\partial}{\partial u}\right) \varphi(p,u) = 0$$
$$\left(u \cdot \frac{\partial}{\partial u} - s\right) \varphi(p,u) = 0$$

Finite-spin/Massless limit of massive equations

Higher-spin massless equations

The massless limit $m \to 0$ of the massive equations (with spin s fixed) is

$$
\begin{aligned}
(\gamma \cdot p)\, \varphi(p,u) &= 0 \\
\left(p \cdot \frac{\partial}{\partial u}\right) \varphi(p,u) &= 0 \\
\left(\gamma \cdot \frac{\partial}{\partial u}\right) \varphi(p,u) &= 0 \\
\left(u \cdot \frac{\partial}{\partial u} - s\right) \varphi(p,u) &= 0
\end{aligned}
$$

which we should supplement with the gauge equivalence relation:

$$\varphi(p,u) \sim \varphi(p,u) + (u \cdot p)\, \varepsilon(p,u)$$

where the gauge parameter obeys similar equations.

Massless equations

In terms of the gauge-invariant distribution

$$\phi(p,u) \;=\; \delta(p\cdot u)\,\varphi(p,u)$$

Finite-spin massless equations

$$\begin{aligned}
&(\gamma\cdot p)\,\phi(p,u)=0\\
&(p\cdot u)\,\phi(p,u)=0,\\
&\left(p\cdot\frac{\partial}{\partial u}\right)\phi(p,u)=0\\
&\left(\gamma\cdot\frac{\partial}{\partial u}\right)\phi(p,u)=0\\
&\left(u\cdot\frac{\partial}{\partial u}-(s-1)\right)\phi(p,u)=0
\end{aligned}$$

Dimensional reduction of massless equations

Finite-spin massless equations in D dimensions

$$(\gamma \cdot p)\,\phi(p,u) = 0$$
$$(p \cdot u)\,\phi(p,u) = 0,$$
$$\left(p \cdot \frac{\partial}{\partial u}\right)\phi(p,u) = 0$$
$$\left(\gamma \cdot \frac{\partial}{\partial u}\right)\phi(p,u) = 0$$
$$\left(u \cdot \frac{\partial}{\partial u} - (s-1)\right)\phi(p,u) = 0$$

Dimensional reduction of massless equations

Finite-spin massless equations in D+1 dimensions

$$\big(\Gamma\cdot P\big)\,\Phi(P,U)=0$$
$$\big(P\cdot U\big)\,\Phi(P,U)=0,$$
$$\left(P\cdot\frac{\partial}{\partial U}\right)\Phi(P,U)=0$$
$$\left(\Gamma\cdot\frac{\partial}{\partial U}\right)\Phi(P,U)=0$$
$$\left(U\cdot\frac{\partial}{\partial U}-(s-1)\right)\Phi(P,U)=0$$

Dimensional reduction of massless equations

Finite-spin massless equations in 5 dimensions

$$\begin{aligned}
&(\Gamma\cdot P)\,\Phi(P,U)=0\\
&(P\cdot U)\,\Phi(P,U)=0,\\
&\left(P\cdot\frac{\partial}{\partial U}\right)\Phi(P,U)=0\\
&\left(\Gamma\cdot\frac{\partial}{\partial U}\right)\Phi(P,U)=0\\
&\left(U\cdot\frac{\partial}{\partial U}-(s-1)\right)\Phi(P,U)=0
\end{aligned}$$

Consider the splittings

$$P_M=(p_\mu,m)\,,\qquad U^M=(u^\mu,v)\,,\qquad \Gamma^M=i\gamma^5(\gamma^\mu,1)\,.$$

Dimensional reduction of massless equations

Finite-spin massive equations in 4 dimensions

$$\begin{aligned}
&(\gamma \cdot p + m\,)\,\Phi = 0\\
&(p \cdot u + mv)\,\Phi = 0\\
&\left(p \cdot \frac{\partial}{\partial u} + m\frac{\partial}{\partial v}\right)\Phi = 0\\
&\left(\gamma \cdot \frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\Phi = 0\\
&\left(u \cdot \frac{\partial}{\partial u} + v\frac{\partial}{\partial v} - (s-1)\right)\Phi = 0
\end{aligned}$$

Infinite-spin/Massless limit of massive equations

Consider the

- change of auxilliary variables

$$\begin{cases} u^\mu = \omega^\mu \, \alpha \\ v = s \, \alpha \end{cases}$$

- change of field

$$\Phi(u,v) = \alpha^{s-1} \Psi(\omega)$$

- limit

$$m \to 0 \,, \qquad s \to \infty \,, \qquad \mu = m \, s \quad \text{fixed}$$

Infinite-spin/Massless limit of massive equations

Infinite-spin massless equations

$$(\gamma \cdot p)\,\Psi(p,\omega) = 0$$

$$(p \cdot \omega + \mu)\,\Psi(p,\omega) = 0$$

$$\left(p \cdot \frac{\partial}{\partial \omega}\right)\Psi(p,\omega) = 0$$

$$\left(\gamma \cdot \frac{\partial}{\partial \omega} + 1\right)\Psi(p,\omega) = 0$$

Infinite-spin/Massless limit of massive equations

Infinite-spin massless equations

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$$\left(p \cdot \frac{\partial}{\partial \omega}\right) \Psi(p,\omega) = 0$$
$$\left(\gamma \cdot \frac{\partial}{\partial \omega} + 1\right) \Psi(p,\omega) = 0$$

Performing a Fourier transform over the auxilliary vector ω leads exactly to Wigner's equations in terms of

$$\tilde{\Psi}(p,\eta) = \int d\omega\ \Psi(p,\omega)\ \exp(-i\,\eta \cdot \omega)$$

Wigner equations

Infinite-spin massless equations (Wigner, 1947)

$$(\gamma \cdot p)\, \widetilde{\Psi}(p,\eta) = 0$$
$$(p \cdot \eta)\, \widetilde{\Psi}(p,\eta) = 0$$
$$\left(p \cdot \frac{\partial}{\partial \eta} - i\,\mu\right) \widetilde{\Psi}(p,\eta) = 0$$
$$(\gamma \cdot \eta + i)\, \widetilde{\Psi}(p,\eta) = 0$$

Action principles

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The example of finite-spin massless fields suggest to make use of a gauge formulation.

Indeed, gauge-invariant action principles corresponding to the helicity representations of the Poincaré group were written for arbitrary integer (Fronsdal, 1978) and half-integer (Fang & Fronsdal, 1978) spin.

This suggests that a formulation *à la* (Fang &) Fronsdal should exist for the continuous-spin representation.

Bosonic action

Fronsdal's finite-spin massless equation

The Fronsdal equation

$$F_{\mu_1\cdots\mu_s} \equiv p^2\,\varphi_{\mu_1\cdots\mu_s} \,-\, p_{(\mu_1}p^\nu\varphi_{\mu_2\cdots\mu_s)\nu} \,+\, p_{(\mu_1}p_{\mu_2}\varphi_{\mu_3\cdots\mu_s)\nu}{}^\nu = 0$$

is the higher-spin generalisation of Klein-Gordon (s=0), Maxwell (s=1) and linearised Ricci (s=2) equations.
It is invariant under the gauge transformations

$$\delta_\varepsilon\varphi_{\mu_1\cdots\mu_s} = p_{(\mu_1}\varepsilon_{\mu_2\cdots\mu_s)}\,, \qquad \varepsilon^\nu{}_{\nu\mu_1\cdots\mu_{s-3}} = 0\,,$$

where the gauge parameter is traceless.
The space of double-traceless

$$\varphi^{\nu\rho}{}_{\nu\rho\mu_1\cdots\mu_{s-4}} = 0$$

and gauge-inequivalent solutions of Fronsdal equations carries the helicity UIR of the Poincaré group $ISO(D-1,1)$ induced from the symmetric tensor representation of the effective little group $SO(D-2)$.

Fronsdal's finite-spin massless equation

Again it turns out to be technically convenient to make use of the generating function

$$\varphi(p,u) = \frac{1}{s!}\varphi_{\mu_1\mu_2\ldots\mu_s}(p)\, u^{\mu_1}\cdots u^{\mu_s}\,.$$

Fronsdal's finite-spin massless equation

Finite-spin massless equation (Fronsdal, 1978)

$$\left[p^2 - (p\cdot u)\left(p\cdot\frac{\partial}{\partial u}\right) + \frac{1}{2}\,(p\cdot u)^2\left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}\right)\right]\varphi(p,u) = 0$$

with the conditions

$$\left(u\cdot\frac{\partial}{\partial u} - s\right)\varphi(p,u) = 0\,, \qquad \left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}\right)^2\varphi(p,u) = 0$$

and gauge equivalence

$$\delta_\varepsilon\varphi(p,u) \;=\; (p\cdot u)\,\varepsilon(p,u)$$

with

$$\left(u\cdot\frac{\partial}{\partial u} - (s-1)\right)\varepsilon(p,u) = 0\,, \qquad \left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}\right)\varepsilon(p,u) = 0$$

Fronsdal-like infinite-spin massless equation

Performing the same steps as before, one may obtain the infinite-spin counterpart of Fronsdal's formulation:

1. remove the homogeneity conditions,
2. perform the following replacement $u \to \omega$, and
3. take into account the rule:

$$p \cdot u \longrightarrow p \cdot \omega + \mu \,, \qquad \frac{\partial}{\partial u} \cdot \frac{\partial}{\partial u} \longrightarrow \frac{\partial}{\partial \omega} \cdot \frac{\partial}{\partial \omega} + 1 \,.$$

Fronsdal-like infinite-spin massless equation

Infinite-spin massless equation (XB & Mourad, 2005)

$$\left[p^2-(p\cdot\omega+\mu)\left(p\cdot\frac{\partial}{\partial\omega}\right)+\frac{1}{2}\,(p\cdot\omega)^2\left(\frac{\partial}{\partial\omega}\cdot\frac{\partial}{\partial\omega}+1\right)\right]\varphi(p,\omega)=0$$

with the conditions

$$\left(\frac{\partial}{\partial\omega}\cdot\frac{\partial}{\partial\omega}+1\right)^2\varphi(p,\omega)=0$$

and gauge equivalence

$$\delta_\varepsilon\varphi(p,\omega)\;=\;(p\cdot\omega+\mu)\,\varepsilon(p,\omega)$$

with

$$\left(\frac{\partial}{\partial\omega}\cdot\frac{\partial}{\partial\omega}+1\right)\varepsilon(p,\omega)=0$$

Fronsdal-like infinite-spin massless equation

Problem:

The Fronsdal equation $F_{\mu_1 \cdots \mu_s} = 0$ is not variational for $s \geqslant 2$. For instance, the Ricci equation is *not* the Euler-Lagrange equation of the Einstein-Hilbert action.

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But the higher-spin generalisation

$$G_{\mu_1\cdots\mu_s}\equiv F_{\mu_1\cdots\mu_s}-\frac{s(s-1)}{2}\,g_{(\mu_1\mu_2}F_{\mu_3\cdots\mu_s)\nu}{}^{\nu}=0$$

of linearised Einstein's equation is variational (Fronsdal, 1978).

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of linearised Einstein's equation is variational (Fronsdal, 1978).

However, this equation blows up in the limit $s \to \infty$, even if one takes into account the infinite rescalings (XB & Mourad, 2005).

Bosonic action

One way out:

Perform Fourier transforms over the auxilliary vector

$$\widetilde{\varepsilon}(p,\eta) = \int d\omega \; \varepsilon(p,\omega)\; \exp(-i\,\eta\cdot\omega)$$

$$\widetilde{\varphi}(p,\eta) = \int d\omega \; \varphi(p,\omega)\; \exp(-i\,\eta\cdot\omega)$$

and solve the tracelessness constraints by distributions.

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and solve the tracelessness constraints by distributions.

In particular, distribution theory states that

$$\big(\eta^2+1\big)\;\widetilde{\varepsilon}(\eta)=0 \quad\Longleftrightarrow\quad \widetilde{\varepsilon}(\eta)=\delta(\eta^2+1)\,\epsilon(\eta)$$

and

$$\big(\eta^2+1\big)^2\;\widetilde{\varphi}(\eta)=0 \quad\Longleftrightarrow\quad \widetilde{\varphi}(\eta)=\delta'(\eta^2+1)\,\Phi(\eta)$$

Bosonic action

In terms of this new field Φ, the Fronsdal-like equation reads

$$\widehat{K}\Phi = 0$$

where the kinetic operator

$$\widehat{K} = -\,\delta(\eta^2+1)\,p^2 + \frac{1}{2}\left(p\cdot\frac{\partial}{\partial\eta} - i\,\mu\right)\delta(\eta^2+1)\left(p\cdot\frac{\partial}{\partial\eta} - i\,\mu\right)$$

is manifestly hermitian, $\widehat{K}^\dagger = \widehat{K}$, with respect to the conjugation

$$\eta^\dagger = \eta\,, \qquad \left(\frac{\partial}{\partial\eta}\right)^\dagger = -\,\frac{\partial}{\partial\eta}\,.$$

Bosonic action

This explains the origin of the bosonic action proposed in (Schuster & Toro, 2014):

$$S[\Phi] = \frac{1}{2}\int d^4x\, d^4\eta\ \Phi(x,\eta)\ \widehat{K}\ \Phi(x,\eta)$$

$$= -\frac{1}{2}\int d^4x\, d^4\eta\ \Phi\left[-\delta'(\eta^2+1)\,\Box + \tfrac{1}{2}\,(\partial_\eta\cdot\partial_x+\mu)\,\delta(\eta^2+1)\,(\partial_\eta\cdot\partial_x+\mu)\right]\Phi$$

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Remarks:

- A similar action principle was proposed by Segal in 2001 for higher-spin massless fields on (anti) de Sitter spacetime.
- The above line of reasoning can be also applied in the fermionic case to construct the action (XB, Setare, Najafizadeh, 2015).
- However, the minimal coupling to external currents lead to current exchanges that do not seem to propagate the correct degrees of freedom although the formal Euclidean version ("Wick rotation") of the action seemingly does (XB, Mourad, Najafizadeh, 2017).

Fermionic action

Fang-Fronsdal-like infinite-spin massless equation

Infinite-spin massless equation (XB & Mourad, 2005)

$$\Big[\gamma\cdot p-(\omega\cdot p+\mu)\,(\gamma\cdot\partial_\omega+1)\Big]\Psi(p,\omega)$$

with the conditions

$$(\gamma\cdot\partial_\omega+1)\,(\partial_\omega\cdot\partial_\omega-1)\;\Psi(p,\omega)=0$$

and gauge equivalence

$$\delta_\varepsilon\varphi(p,\omega)\;=\;(p\cdot\omega+\mu)\,\varepsilon(p,\omega)$$

with

$$(\gamma\cdot\partial_\omega+1)\;\varepsilon(p,\omega)=0$$

Fermionic action

Perform Fourier transforms over the auxilliary vector

$$\widetilde{\varepsilon}(p,\eta) = \int d\omega \ \varepsilon(p,\omega) \ \exp(-i\,\eta\cdot\omega)$$

$$\widetilde{\psi}(p,\eta) = \int d\omega \ \psi(p,\omega) \ \exp(-i\,\eta\cdot\omega)$$

and solve the tracelessness constraints by distributions.

Fermionic action

Perform Fourier transforms over the auxilliary vector

$$\widetilde{\varepsilon}(p,\eta) = \int d\omega\ \varepsilon(p,\omega)\ \exp(-i\,\eta\cdot\omega)$$

$$\widetilde{\psi}(p,\eta) = \int d\omega\ \psi(p,\omega)\ \exp(-i\,\eta\cdot\omega)$$

and solve the tracelessness constraints by distributions.

Distribution theory allows to show that

$$(\gamma\cdot\eta - i)\ \widetilde{\varepsilon}(p,\eta) = 0 \quad \Longleftrightarrow \quad \widetilde{\varepsilon}(p,\eta) = \delta(\eta^2+1)\,(\gamma\cdot\eta + i\,)\,\epsilon(p,\eta)$$

and

$$(\gamma\cdot\eta - i\,)\,(\eta^2+1)\ \widetilde{\psi}(p,\eta) = 0 \Longleftrightarrow \widetilde{\psi}(p,\eta) = \delta'(\eta^2+1)\big(\gamma\cdot\eta + i\,\big)\Psi(p,\eta)$$

Fermionic action

In terms of the new field Ψ, the Fang-Fronsdal-like equation reads

$$\widehat{K}\Psi = 0$$

where the kinetic operator

$$\widehat{K} = i\,\delta'(\eta^2+1)\big(\gamma\cdot\eta - i\,\big)(\gamma\cdot p) + i\,\delta(\eta^2+1)\left(p\cdot\frac{\partial}{\partial\eta} - i\mu\right)$$

satisfies $\widehat{K}^\dagger = \gamma^0\,\widehat{K}\,\gamma^0$ as does the Dirac kinetic operator.

Fermionic action

This suggest the action (XB, Setare, Najafizadeh, 2015):

$$S_{free} = \int d^4x\, d^4\eta\; \overline{\Psi}\; \widehat{\mathcal{K}}\; \Psi$$

$$= \int d^4x\, d^4\eta\; \overline{\Psi} \Big[\delta'(\eta^2+\sigma)\big(\gamma\cdot\eta - i\,\sigma\big)(\gamma\cdot\partial_x) + \delta(\eta^2+\sigma)\big(\partial_\eta\cdot\partial_x + \sigma\mu\big)\Big]\,\Psi\,,$$

where $\overline{\Psi} = \Psi^\dagger\,\gamma^0$.

Summary of results and open problems

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1. Wigner's exotic representations & equations from infinite-spin/massless limit of massive
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     - bosons & fermions (Metsaev, 2016) also on anti de Sitter spacetime
4. Cubic interactions with matter fields
   - Scalar matter, covariant formulation (XB, Mourad, Najafizadeh, 2017)
   - Exhaustive classification, light-cone formulation (Metsaev, 2017)

List of some open problems

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4. Nonvanishing cosmological constant
   - do they $\exists$ as (unitary) irreps of (anti) de Sitter isometry groups $SO(D,1)$ or $SO(D-1,2)$?
   - comparison with the action principles of (Metsaev, 2016)
   - does there $\exists$ an exotic nontrivial flat limit of higher-spin algebra? of higher-spin gravity?