Proof of the Weak Gravity Conjecture from Black Hole Entropy

Grant N. Remmen

Berkeley Center for Theoretical Physics
Miller Institute for Basic Research in Science
University of California, Berkeley

arXiv:1801.08546 with Cliff Cheung & Junyu Liu

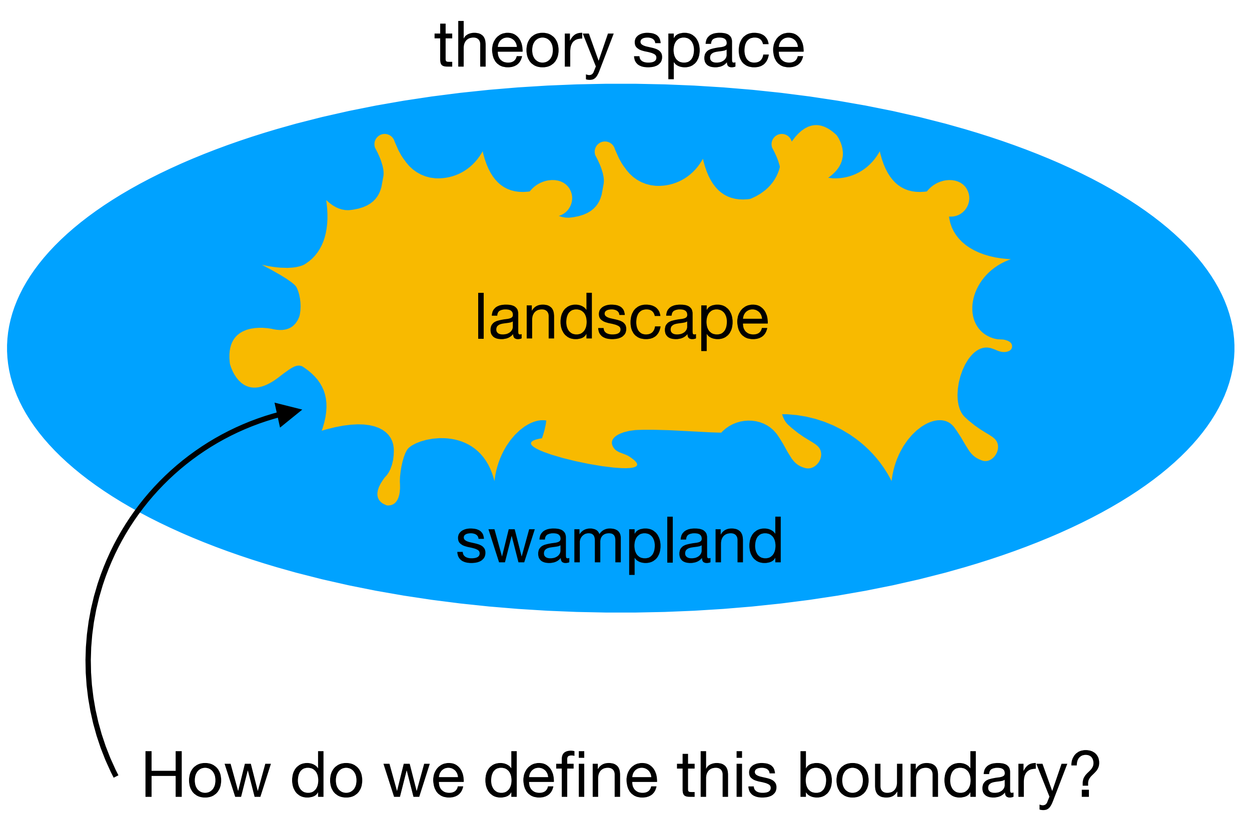
Gravity, Cosmology & Physics Beyond the Standard Model
June 2018, Paris

BERKELEY CENTER FOR
THEORETICAL PHYSICS

MILLER
INSTITUTE

Introduction:
The swampland program

The swampland program



Landscape: Set of EFTs consistent with UV completion in quantum gravity

Swampland: Set of EFTs **in**consistent with UV completion in quantum gravity

String theory provides a large number of consistent vacua, with different sets of low-energy laws of physics. But it does not completely populate the space of all possible EFTs.

The swampland program

How to find the boundary of the landscape?

1. Observe examples in string theory and make conjectures
    - Example: Weak Gravity Conjecture Arkani-Hamed et al. [hep-th/0601001]

2. Prove bounds from infrared physics principles
    - Unitarity
    - Causality
    - Analyticity
    - Examples:
        - Einstein-Maxwell theory Cheung, GNR [1407.7865]
        - Higher-curvature gravity (R^2, R^4 terms) Bellazzini, Cheung, GNR [1509.00851]; Cheung, GNR [1608.02942]
        - Massive gravity Cheung, GNR [1601.04068]
        - $(\partial\phi)^4$ and F^4 couplings Adams et al. [hep-th/0602178]

This paper:

Prove the Weak Gravity Conjecture using a new IR argument related to black hole entropy.

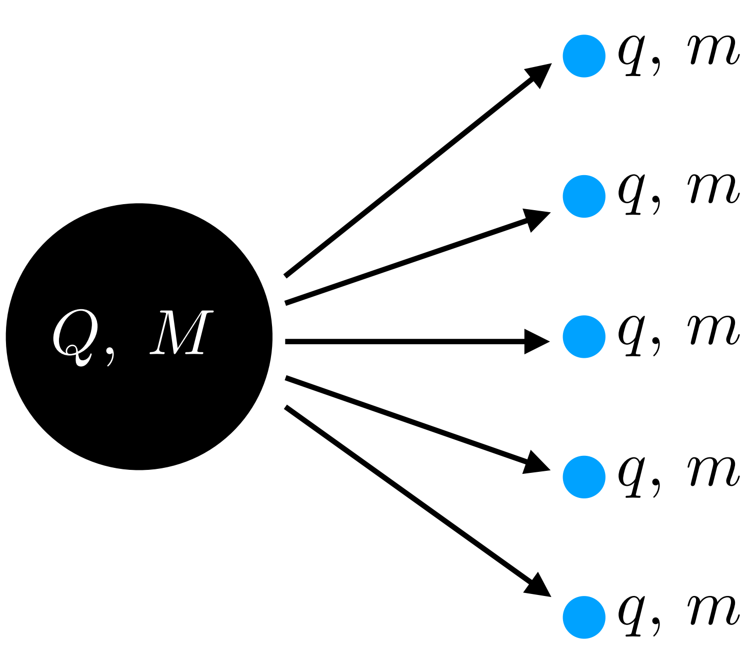
Introduction: The Weak Gravity Conjecture

- An ultraviolet consistency condition for quantum gravity.

- Statement: For any $U(1)$ gauge theory coupled consistently with quantum gravity, there must exist in the spectrum a state with charge q and mass m such that

$$\frac{q}{m} > \frac{1}{m_{\mathrm{Pl}}}$$

- Thus, "gravity is the weakest force".



Original justification: Arkani-Hamed et al. [hep-th/0601001]

A black hole of charge Q and mass M can only decay into states satisfying

$$\frac{q}{m} > \frac{Q}{M}$$

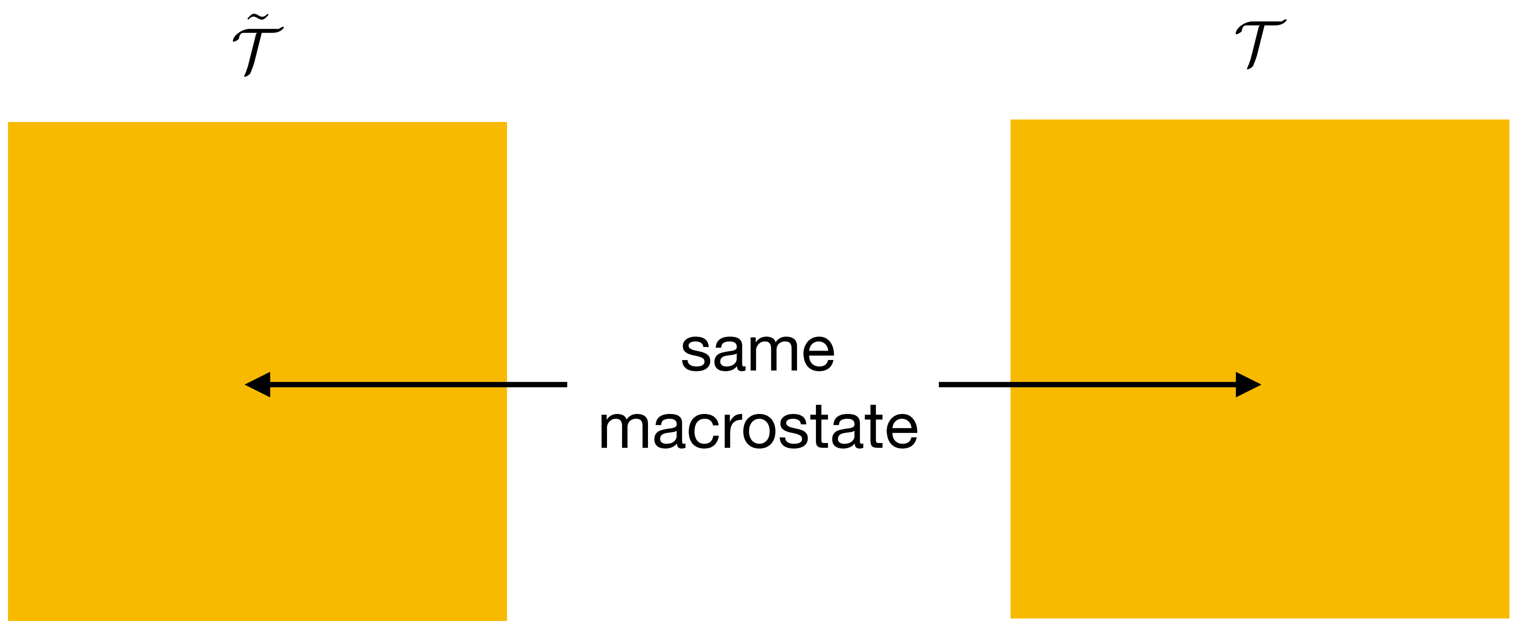
Extremal BH decay $\Longrightarrow$ WGC

Why BH decay? BH remnant pathologies

Thermodynamics thought experiment

Comparing systems

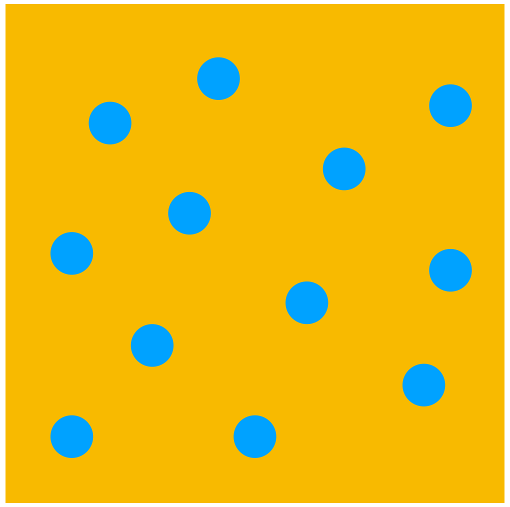
Consider two systems:



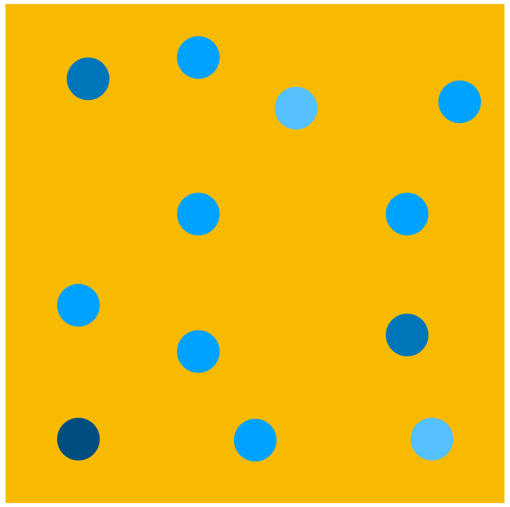
Comparing systems

Consider two systems:

System without extra microstates



System with extra microstates



Entropy: $\tilde{S}$ S

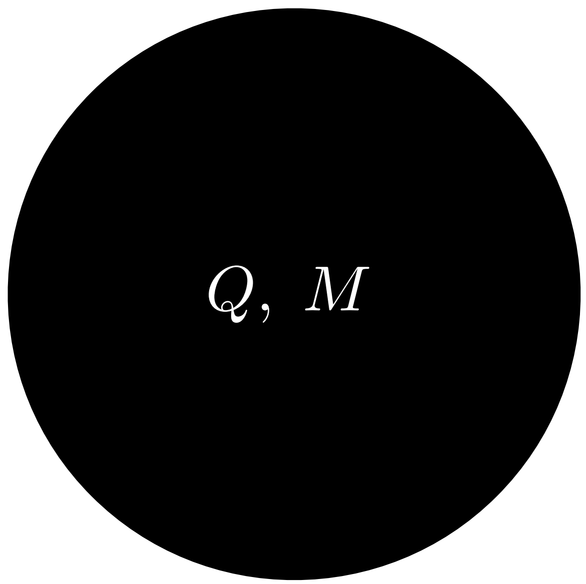
System with extra modes has greater entropy:

$$\Delta S = S - \tilde{S} > 0$$

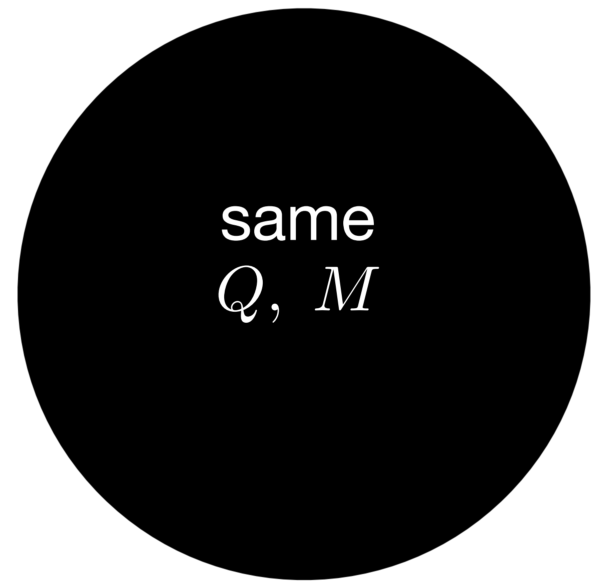
Black hole entropy comparison

We can compute the black hole's entropy in two situations:

Theory $\tilde{\mathcal{L}}$ without higher-derivative terms



Theory $\mathcal{L} = \tilde{\mathcal{L}} + \Delta\mathcal{L}$ with higher-derivative terms



Present in theory in UV:
Massive states that generated higher-curvature terms

Integrated out to generate EFT

Compare entropy in the two theories:

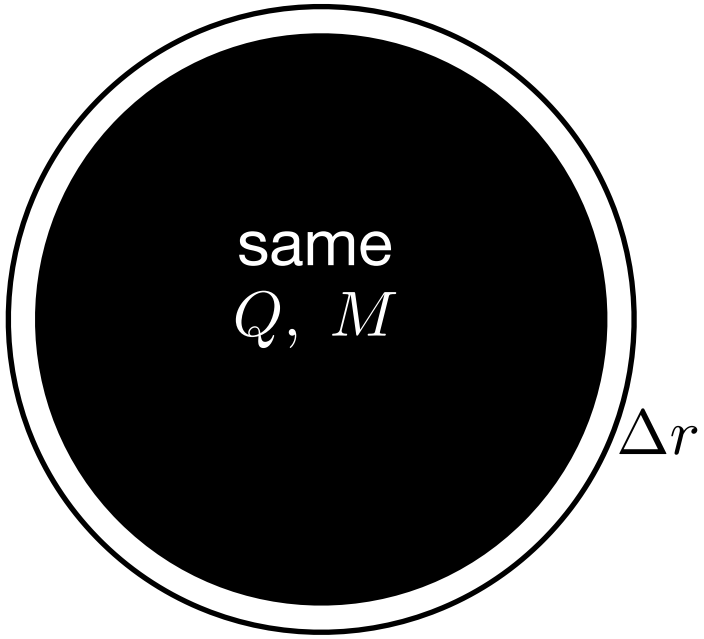
$$\boxed{\Delta S = S - \tilde{S} > 0}$$

Black hole entropy comparison

We can compute the black hole's entropy in two situations:

Pure Einstein-Maxwell theory

IR EFT



Area $\tilde{A}$ dictated by Einstein equation

Entropy $\tilde{S} = \tilde{A}/4G$

Area $A = \tilde{A} + \Delta A$ dictated by higher-derivative-corrected Einstein equation

Entropy given by Wald's formula:

$$S = -2\pi \int_{\mathrm{H}} \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma}$$

Einstein-Maxwell effective action

Pure Einstein-Maxwell theory

$$\widetilde{\mathcal{L}} = \frac{1}{2\kappa^2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

IR EFT

$$\mathcal{L} = \tilde{\mathcal{L}} + \Delta\mathcal{L}$$

$$\boxed{\begin{aligned} \Delta\mathcal{L} = \quad & c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\ & + c_4 R F_{\mu\nu} F^{\mu\nu} + c_5 R_{\mu\nu} F^{\mu\rho} F^{\nu}{}_{\rho} + c_6 R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \\ & + c_7 F_{\mu\nu} F^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + c_8 F_{\mu\nu} F^{\nu\rho} F_{\rho\sigma} F^{\sigma\mu} \end{aligned}}$$

Other types of terms that we can drop:

- All terms involving $\nabla_\rho F_{\mu\nu}$, since the Bianchi identities allow us to write these in terms of $\nabla_\mu F^{\mu\nu} = 0$ and terms already included
- Dimension-independent total derivatives

Einstein-Maxwell effective action

Pure Einstein-Maxwell theory

$$\widetilde{\mathcal{L}} = \frac{1}{2\kappa^2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

IR EFT

$$\mathcal{L} = \tilde{\mathcal{L}} + \Delta\mathcal{L}$$

$$\boxed{\begin{aligned} \Delta\mathcal{L} = \quad & c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\ & + c_4 R F_{\mu\nu} F^{\mu\nu} + c_5 R_{\mu\nu} F^{\mu\rho} F^{\nu}{}_{\rho} + c_6 R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \\ & + c_7 F_{\mu\nu} F^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + c_8 F_{\mu\nu} F^{\nu\rho} F_{\rho\sigma} F^{\sigma\mu} \end{aligned}}$$

We will prove a positivity bound on a combination of the c_i.

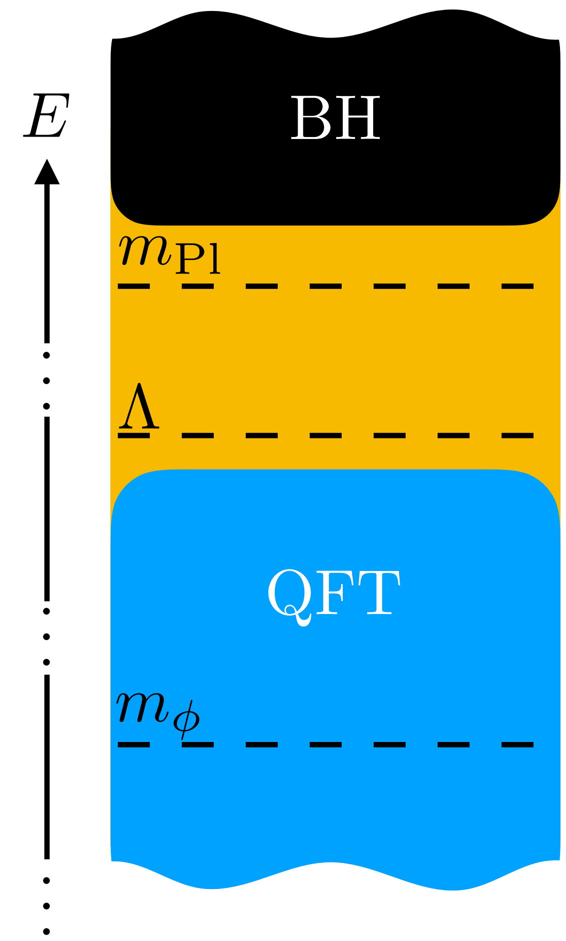
We'll then demonstrate, surprisingly, that this bound precisely implies the Weak Gravity Conjecture.

Proof of $\Delta S > 0$

Assumptions

For the purposes of this proof, we assume:

1. There exist quantum fields ϕ at a mass scale m_ϕ satisfying
$$m_\phi \ll \Lambda,$$
where Λ is the scale at which QFT breaks down.
In general, Λ can be much smaller than the Planck scale.



Assumptions

For the purposes of this proof, we assume:

1. There exist quantum fields ϕ at a mass scale m_ϕ satisfying
$$m_\phi \ll \Lambda,$$
where Λ is the scale at which QFT breaks down.
In general, Λ can be much smaller than the Planck scale.

2. The fields ϕ couple to photons and gravitons so that the higher-dimension operators are generated at tree level, e.g., $\sim \phi R,\ \phi F^2$
so:
$$\underbrace{c_i \propto 1/m_\phi^2}_{\text{QFT effect}} \gg \underbrace{1/\Lambda^2}_{\text{Quantum gravity “slop”}}$$
Couplings like this are common in string theory: dilaton and moduli are massless in supersymmetric limit, and acquire masses if SUSY is broken.

3. We will consider black holes with charge large enough that the specific heat is positive. As we'll see, this will be necessary for our Euclidean path integral argument.

Euclidean path integral

Positively charged black hole, charge Q and mass M, spacetime dimension D

Perturbed metric $g_{\mu\nu} = \widetilde{g}_{\mu\nu} + \Delta g_{\mu\nu}$ computed from
perturbed Lagrangian $\mathcal{L} = \widetilde{\mathcal{L}} + \Delta\mathcal{L}$

Inverse temperature of perturbed BH,

$$\beta = \partial_M S = \widetilde{\beta} + \Delta\beta,$$

defines periodicity in Euclidean time for the Euclidean path integral,

$$\boxed{e^{-\beta F(\beta)} = Z(\beta) = \int d[\hat{g}]d[\hat{A}]\, e^{-I[\hat{g},\hat{A}]}}$$

where

$I = \widetilde{I} + \Delta I$ is the Euclidean action
(spacetime integral of Wick-rotated Lagrangian)

$F(\beta)$ is the free energy

$\hat{g}, \hat{A}$ are integration variables for the metric and gauge field

Euclidean path integral

Positively charged black hole, charge Q and mass M, spacetime dimension D

Perturbed metric $g_{\mu\nu} = \widetilde{g}_{\mu\nu} + \Delta g_{\mu\nu}$ computed from perturbed Lagrangian $\mathcal{L} = \widetilde{\mathcal{L}} + \Delta\mathcal{L}$

Ultraviolet completion: introduce integration variable $\hat{\phi}$ for the heavy fields that are integrated out when we go from UV to IR:

$$\int d[\hat{g}]d[\hat{A}]d[\hat{\phi}]\, e^{-I_{\mathrm{UV}}[\hat{g},\hat{A},\hat{\phi}]} = \int d[\hat{g}]d[\hat{A}]\, e^{-I[\hat{g},\hat{A}]}$$

We define the vev of $\hat{\phi}$ to be zero in flat space.

For the on-shell black hole in the $\mathcal{L}$ theory, $\phi \neq 0$, since equations of motion dictate $\phi \sim R, F^2$

Going off shell

We can evaluate the Euclidean action at any field configuration we wish, including one that does *not* satisfy the classical equations of motion.

In particular, let's evaluate $I_{\rm UV}$ at $\hat{\phi} = 0$, which turns off all the higher-dimension operators in $\Delta\mathcal{L}$, so we have the simple mathematical fact:

$$\boxed{I_{\rm UV}[\hat{g}, \hat{A}, 0] = \widetilde{I}[\hat{g}, \hat{A}]}$$

where $\widetilde{I}$ is the Euclidean action for pure Einstein-Maxwell theory.

This observation will allow us to compare the two black hole entropies in $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ via an argument that only involves working in a *single* theory.

Free energy inequality

Putting our thermodynamic argument together, we have the string of (in)equalities relating the free energies of an Einstein-Maxwell and perturbed Reissner-Nordström black hole at the same temperature:

$$-\log Z(\beta) = I_{\mathrm{UV}}[g_\beta, A_\beta, \phi_\beta] \longleftarrow$$

by saddle-point approximation where $g_\beta, A_\beta, \phi_\beta$ are the solutions to classical EoM in UV theory, with periodicity β

Free energy inequality

Putting our thermodynamic argument together, we have the string of (in)equalities relating the free energies of an Einstein-Maxwell and perturbed Reissner-Nordström black hole at the same temperature:

$$-\log Z(\beta) = I_{\mathrm{UV}}[g_\beta, A_\beta, \phi_\beta] \longleftarrow$$ by saddle-point approximation

$$< I_{\mathrm{UV}}[\widetilde{g}_\beta, \widetilde{A}_\beta, 0] \longleftarrow$$ if the extremum is a local minimum (will discuss shortly)

Free energy inequality

Putting our thermodynamic argument together, we have the string of (in)equalities relating the free energies of an Einstein-Maxwell and perturbed Reissner-Nordström black hole at the same temperature:

$$
\begin{aligned}
-\log Z(\beta) &= I_{\mathrm{UV}}[g_\beta, A_\beta, \phi_\beta] && \longleftarrow \text{ by saddle-point approximation} \\
&< I_{\mathrm{UV}}[\widetilde{g}_\beta, \widetilde{A}_\beta, 0] && \longleftarrow \text{ if the extremum is a local minimum} \\
&= \widetilde{I}[\widetilde{g}_\beta, \widetilde{A}_\beta] && \longleftarrow \text{ by the off-shell relation we found previously, relating } I_{\mathrm{UV}} \text{ and } \widetilde{I}
\end{aligned}
$$

Free energy inequality

Putting our thermodynamic argument together, we have the string of (in)equalities relating the free energies of an Einstein-Maxwell and perturbed Reissner-Nordström black hole at the same temperature:

$$\begin{aligned}
-\log Z(\beta) &= I_{\mathrm{UV}}[g_\beta, A_\beta, \phi_\beta] &&\longleftarrow \text{ by saddle-point approximation} \\
&< I_{\mathrm{UV}}[\widetilde{g}_\beta, \widetilde{A}_\beta, 0] &&\longleftarrow \text{ if the extremum is a local minimum} \\
&= \widetilde{I}[\widetilde{g}_\beta, \widetilde{A}_\beta] &&\longleftarrow \text{ by the off-shell relation} \\
&= -\log \widetilde{Z}(\beta) &&\longleftarrow \text{ by saddle-point approximation, again}
\end{aligned}$$

Now, $\log \widetilde{Z}(\beta)$ does *not* correspond to the free energy of a pure Reissner-Nordström black hole of mass M, since β is the *perturbed* inverse temperature ($\neq \widetilde{\beta}$). To account for this, we have

$$\begin{aligned}
\log \widetilde{Z}(\beta) &= \log \widetilde{Z}(\widetilde{\beta}) + \Delta\beta\,\partial_{\widetilde{\beta}} \log \widetilde{Z}(\widetilde{\beta}) \\
&= \log \widetilde{Z}(\widetilde{\beta}) - M\partial_M \Delta S &&\longleftarrow \text{ using } M = -\partial_{\widetilde{\beta}} \log \widetilde{Z}(\widetilde{\beta}) \text{ and } \Delta\beta = \partial_M \Delta S
\end{aligned}$$

Free energy inequality

$$\log \widetilde{Z}(\beta) = \log \widetilde{Z}(\widetilde{\beta}) - M\partial_M \Delta S$$

By the definition of free energy in the canonical ensemble,

$$\log Z(\beta) = S - \beta M = (1 - M\partial_M)S$$
$$\log \widetilde{Z}(\widetilde{\beta}) = \widetilde{S} - \widetilde{\beta} M = (1 - M\partial_M)\widetilde{S}$$

Using the above expressions and reshuffling terms, our inequality

$$-\log Z(\beta) < -\log \widetilde{Z}(\beta)$$

i.e., $F(\beta) < \widetilde{F}(\beta)$, becomes

$$\boxed{\Delta S > 0}$$

Minimization of the Euclidean action

- We needed the saddle point, corresponding to the classical solution, to be a local minimum. Equivalently, we needed the Euclidean action to be stable under small off-shell perturbations.

- What about conformal saddle-point instabilities? These have been shown to be gauge artifacts. Gibbons, Hawking, Perry (1978); Gibbons, Perry (1978)

- The Euclidean Schwarzschild black hole is known to have a bona fide instability. Gross, Perry, Yaffe (1982)

- However, this instability is always connected with negative specific heat. Prestidge [hep-th/9907163]; Reall [hep-th/0104071]; Monteiro, Santos [0812.1767]

- For large enough charge, the specific heat of the black hole is positive. In $D = 4$, this requires $q/m > \sqrt{3}/2$ in natural units. Hereafter, we'll focus on black holes where this is satisfied.

Example UV completion

Let's see how this works in a particular example:
Higher-dimension operators completed by a massive scalar field ϕ

Euclidean action for the UV completion:

$$I_{\mathrm{UV}}[g, A, \phi] = \int d^D x \sqrt{g} \bigg[-\frac{1}{2\kappa^2} R + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ + \left(\frac{a_\phi}{\kappa} R + b_\phi \kappa F_{\mu\nu} F^{\mu\nu} \right) \phi + \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi + \frac{1}{2} m_\phi^2 \phi^2 \bigg]$$

Equation of motion for ϕ:

$$\phi = \frac{1}{\nabla^2 - m_\phi^2} \left(\frac{a_\phi}{\kappa} R + b_\phi \kappa F_{\mu\nu} F^{\mu\nu} \right)$$

Low-energy effective theory:

$$I[g, A] = \int d^D x \sqrt{g} \left[-\frac{1}{2\kappa^2} R + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2m_\phi^2} \left(\frac{a_\phi}{\kappa} R + b_\phi \kappa F_{\mu\nu} F^{\mu\nu} \right)^2 \right]$$

Example UV completion

Euclidean action for the UV completion:

$$I_{\mathrm{UV}}[g,A,\phi]=\int d^Dx\sqrt{g}\Big[-\frac{1}{2\kappa^2}R+\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
$$+\left(\frac{a_\phi}{\kappa}R+b_\phi\kappa F_{\mu\nu}F^{\mu\nu}\right)\phi+\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi+\frac{1}{2}m_\phi^2\phi^2\Big]$$

Low-energy effective theory:

$$I[g,A]=\int d^Dx\sqrt{g}\left[-\frac{1}{2\kappa^2}R+\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2m_\phi^2}\left(\frac{a_\phi}{\kappa}R+b_\phi\kappa F_{\mu\nu}F^{\mu\nu}\right)^2\right]$$

Off-shell UV action:

$$I_{\mathrm{UV}}[g,A,0]=\int d^Dx\sqrt{g}\left[-\frac{1}{2\kappa^2}R+\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]=\widetilde{I}[g,A]$$

Thus, we have

$$\boxed{I[g,A]<I[\widetilde{g},\widetilde{A}]<\widetilde{I}[\widetilde{g},\widetilde{A}]}$$

Unitarity and monotonicity

- In the explicit example, signs of the couplings depended on the absence of ghosts or tachyons in the UV completion, so $\Delta S > 0$ is connected with unitarity. Connections with other IR consistency bounds on couplings derived from unitarity and analyticity?

- Connection with monotonicity theorems for RG flows: If spectrum is hierarchical, we can apply our logic for $\Delta S > 0$ to each mass threshold, one at a time. Then:

$$\boxed{\Delta S = \int_{\mathrm{UV}}^{\mathrm{IR}} dS}$$

  where $dS > 0$ for each state.

- Conjecture: Differential entropy shift may continue to be positive at the quantum (i.e., loop) level, giving us a monotonic function along RG flow.

Classical vs. quantum

Leading contributions

Let's define some rescaled couplings for convenience:

$$d_{1,2,3} = \kappa^2 c_{1,2,3}, \qquad d_{4,5,6} = c_{4,5,6}, \qquad d_{7,8} = \kappa^{-2} c_{7,8}$$

Example tree-level completion:
Scalar ϕ couples to curvature and gauge field as $\sim \phi R/\kappa$, $\sim \kappa\phi F^2$

Contributions to higher dimension operators:

- Tree level: $\delta(d_i) \sim \dfrac{1}{m_\phi^2}$ from the propagator

- Loop level:

    - Renormalization of Newton's constant: $\delta(\kappa^{-2}) \sim m_\phi^{D-2}$

    - Loop-level completions of the gravitational higher-dimension operators: $\delta(d_i) \sim \kappa^2 m_\phi^{D-4}$

Leading contributions

Let's define some rescaled couplings for convenience:

$$d_{1,2,3} = \kappa^2 c_{1,2,3}, \qquad d_{4,5,6} = c_{4,5,6}, \qquad d_{7,8} = \kappa^{-2} c_{7,8}$$

Example tree-level completion:
Scalar ϕ couples to curvature and gauge field as $\sim \phi R/\kappa$, $\sim \kappa\phi F^2$

Contributions to higher dimension operators:

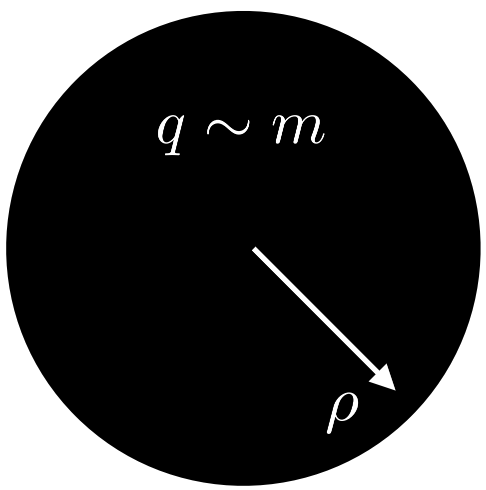
- Tree level: $\delta(d_i) \sim \dfrac{1}{m_\phi^2}$ from the propagator

- Loop level:

    - Gauge interactions contribute similarly, but enhanced by the charge-to-mass ratio of the fundamental charged particles.
    - If these particles satisfy the WGC, we're already done, so let's conservatively assume the particles fail or marginally satisfy the WGC.

Region of interest

Estimating the sizes of the entropy corrections for a black hole:



$$S \sim \frac{\rho^{D-2}}{\kappa^2} + \rho^{D-2} m_\phi^{D-2} + \rho^{D-4} m_\phi^{D-4} + \frac{\rho^{D-4}}{\kappa^2 m_\phi^2} + \cdots$$

Bekenstein-Hawking entropy

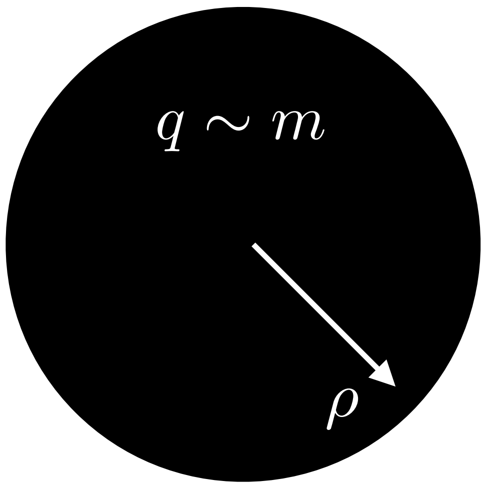
Loop correction to G

Loop contribution to $\Delta\mathcal{L}$

Tree contribution to $\Delta\mathcal{L}$

Region of interest

Estimating the sizes of the entropy corrections for a black hole:



$$S \sim \frac{\rho^{D-2}}{\kappa^2} + \rho^{D-2} m_\phi^{D-2} + \rho^{D-4} m_\phi^{D-4} + \frac{\rho^{D-4}}{\kappa^2 m_\phi^2} + \cdots$$

Tree contribution to $\Delta\mathcal{L}$ (4th term) dominates over all quantum (i.e., loop) corrections (2nd and 3rd terms), provided:

$$\boxed{\rho \ll \frac{1}{\kappa m_\phi^{D/2}}}$$

This is consistent with the regime of validity of the EFT, $\rho \gg 1/m_\phi$, since we take $m_\phi \ll m_{\mathrm{Pl}}$. We will therefore consider black holes in this size range.

Black hole spacetime

The black hole system

Macrostate: Charged black hole in $D = 4$ spacetime dimensions with charge Q and mass M measured at spatial infinity

Komar formalism:

$$Q = -\int_{i^0} \mathrm{d}^{D-2}\Omega_{D-2}\sqrt{\gamma}n_\mu \nabla_\nu F^{\mu\nu}$$

$$\frac{D-3}{D-2}\kappa^2 M = \int_{i^0} \mathrm{d}^{D-2}\Omega_{D-2}\sqrt{\gamma}n_\mu \sigma_\nu \nabla^\mu K^\nu$$

Convenient units:

$$m = \frac{\kappa^2 M}{8\pi}$$

$$q = \frac{\kappa Q}{4\sqrt{2}\pi}$$

$$\kappa^2 = 8\pi G$$

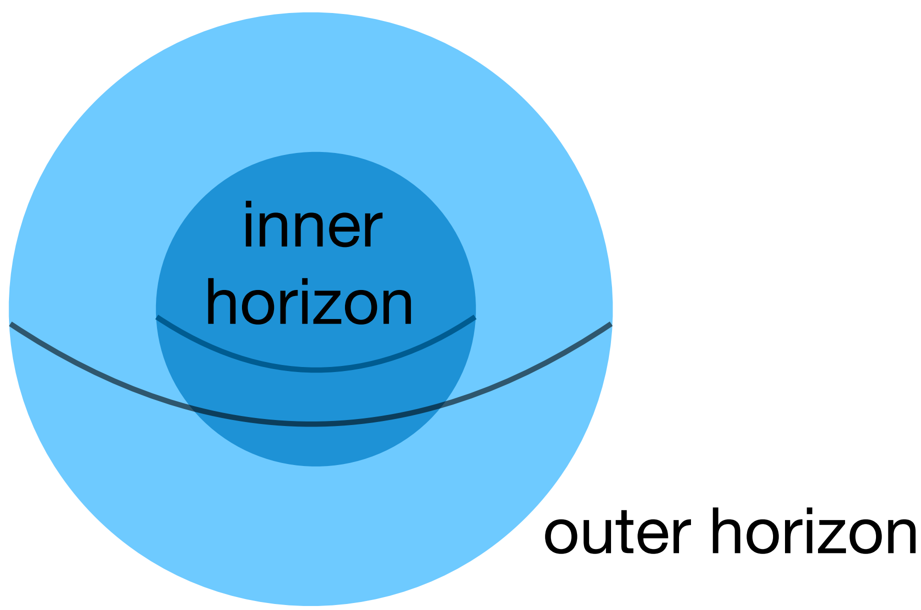
Charge-to-mass parameter:

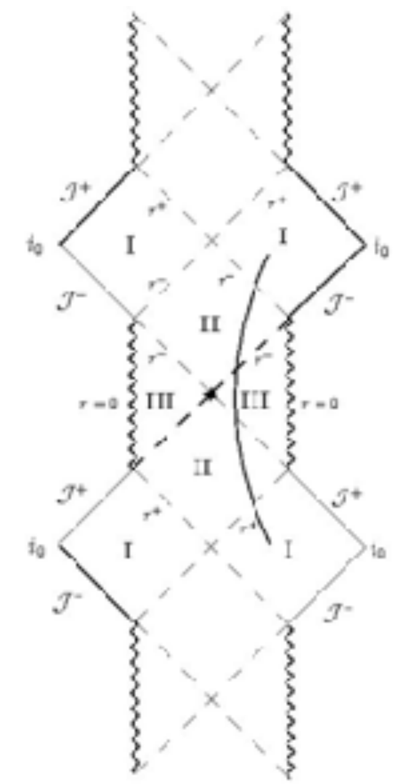
$$\xi = \sqrt{1 - \frac{q^2}{m^2}}$$

$\xi = 0 \implies$ extremal
$\xi = 1 \implies$ uncharged
$\xi = 1/2 \implies q/m = \sqrt{3}/2$

Unperturbed solution

The Reissner-Nordström black hole:





Static, spherically-symmetric metric:

$$\mathrm{d}s^2 = \widetilde{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\widetilde{f}(r)\mathrm{d}t^2 + \frac{1}{\widetilde{g}(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

Unperturbed components ($D = 4$):

$$\widetilde{f}(r) = \widetilde{g}(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}$$

Field strength:

$$\widetilde{F} = \frac{Q}{4\pi r^2}\mathrm{d}t \wedge \mathrm{d}r$$

Outer (event) horizon:

$$r = \widetilde{\rho}$$

$$\widetilde{\rho} = m + \sqrt{m^2 - q^2} = m(1 + \xi)$$

Extremality condition:

$$\frac{q}{m} \leq 1$$

Perturbed charged black hole metric

Need to calculate the change in area of the black hole of fixed Q, M due to the higher-dimension operators Kats, Motl, Padi [hep-th/0606100]

From definition of Ricci tensor and spherically-symmetric metric:

$$g(r) = 1 - \frac{\kappa^2 M}{4\pi r} - \frac{1}{r}\int_r^{+\infty} \mathrm{d}r\; r^2 \left(\frac{R^t{}_t - R^r{}_r}{2} - R^i{}_i\right)$$

$$f(r) = g(r) \exp\left[\int_r^{+\infty} \mathrm{d}r \frac{r}{g(r)} (R^t{}_t - R^r{}_r)\right]$$

Inputting Einstein equation,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \qquad T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_{\mathrm{mat}})}{\delta g^{\mu\nu}}$$

can rewrite as

$$g(r) = 1 - \frac{\kappa^2 M}{4\pi r} - \frac{\kappa^2}{r}\int_r^{+\infty} \mathrm{d}r\; r^2 T^t{}_t$$

$$f(r) = g(r) \exp\left[\kappa^2 \int_r^{+\infty} \mathrm{d}r \frac{r}{g(r)} (T^t{}_t - T^r{}_r)\right]$$

The corrected energy-momentum tensor

For now, focus on computing the radial metric component g

Need to find the corrected energy $T^t{}_t$

Background:

$$\widetilde{T}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\widetilde{\mathcal{L}}_{\text{mat}})}{\delta g^{\mu\nu}} = F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

Treat higher-dimension operators as perturbation to background energy-momentum tensor

$$\Delta T_{\mu\nu} = \Delta T^{(g)}_{\mu\nu} + \Delta T^{(F)}_{\mu\nu}$$

Contribution from $g_{\mu\nu}$
equation of motion
(correction to Einstein's equations)

Contribution from A_μ
equation of motion
(correction to Maxwell's equations)

The corrected energy-momentum tensor

Metric part of corrected energy-momentum:

$$
\begin{aligned}
\Delta T^{(g)}_{\mu\nu} = & -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\Delta\mathcal{L})}{\delta g^{\mu\nu}} \\
= & \; c_1\left(g_{\mu\nu}R^2 - 4RR_{\mu\nu} + 4\nabla_\nu\nabla_\mu R - 4g_{\mu\nu}\Box R\right) \\
& + c_2\left(g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma} + 4\nabla_\rho\nabla_\nu R_\mu{}^\rho - 2\Box R_{\mu\nu} - g_{\mu\nu}\Box R - 4R_\mu{}^\rho R_{\rho\nu}\right) \\
& + c_3\left(g_{\mu\nu}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} - 8\Box R_{\mu\nu} + 4\nabla_\nu\nabla_\mu R \right. \\
& \qquad \left. + 8R_\mu{}^\rho R_{\rho\nu} - 8R^{\alpha\beta}R_{\mu\alpha\nu\beta}\right) \\
& + c_4\left[g_{\mu\nu}RF_{\rho\sigma}F^{\rho\sigma} - 4RF_\mu{}^\rho F_{\nu\rho} - 2F_{\rho\sigma}F^{\rho\sigma}R_{\mu\nu} + 2\nabla_\mu\nabla_\nu F_{\rho\sigma}F^{\rho\sigma} - 2g_{\mu\nu}\Box(F_{\rho\sigma}F^{\rho\sigma})\right] \\
& + c_5\left[g_{\mu\nu}R^{\alpha\beta}F_{\alpha\rho}F_\beta{}^\rho - 4R_{\nu\sigma}F_{\mu\rho}F^{\sigma\rho} - 2R^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu} - g_{\mu\nu}\nabla_\alpha\nabla_\beta(F^\alpha{}_\rho F^{\beta\rho}) \right. \\
& \qquad \left. + 2\nabla_\alpha\nabla_\nu(F_{\mu\beta}F^{\alpha\beta}) - \Box(F_{\mu\rho}F_\nu{}^\rho)\right] \\
& + c_6\left[g_{\mu\nu}R^{\rho\sigma\alpha\beta}F_{\rho\sigma}F_{\alpha\beta} - 6F_{\alpha\nu}F^{\beta\gamma}R^\alpha{}_{\mu\beta\gamma} - 4\nabla_\beta\nabla_\alpha(F^\alpha{}_\mu F^\beta{}_\nu)\right] \\
& + c_7\left[g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta} - 8F_{\alpha\beta}F^{\alpha\beta}F_\mu{}^\rho F_{\nu\rho}\right] \\
& + c_8\left(g_{\mu\nu}F_{\alpha\beta}F^{\beta\gamma}F_{\gamma\delta}F^{\delta\alpha} - 8F_{\mu\alpha}F^{\alpha\beta}F_{\beta\gamma}F^\gamma{}_\nu\right)
\end{aligned}
$$

To linear order in c_i, input background Reissner-Nordström solution

The corrected energy-momentum tensor

Corrected Maxwell's equations:

$$\begin{aligned}\nabla_\nu F^{\mu\nu} = \quad & 4c_4 \nabla_\nu (R F^{\mu\nu}) \\ & + 2c_5 \nabla_\nu (R^{\mu\rho} F_\rho{}^\nu - R^{\nu\rho} F_\rho{}^\mu) \\ & + 4c_6 \nabla_\nu (R^{\mu\nu\rho\sigma} F_{\rho\sigma}) \\ & + 8c_7 \nabla_\nu (F^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}) \\ & + 8c_8 \nabla_\nu (F^{\mu\rho} F_{\rho\sigma} F^{\nu\sigma}) \\ = \quad & \nabla_\nu (\Delta F^{\mu\nu})\end{aligned}$$

Gauge field part of corrected energy-momentum:

$$\Delta T^{(F)}_{\mu\nu} = F_{\mu\rho} \Delta F_\nu{}^\rho + F_\nu{}^\rho \Delta F_{\mu\rho} - \frac{1}{2} g_{\mu\nu} F_{\rho\sigma} \Delta F^{\rho\sigma}$$

To linear order in c_i, input background Reissner-Nordström solution

Perturbed solution

- General form of the metric:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -f(r)\mathrm{d}t^2 + \frac{1}{g(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

- Putting everything together, we can compute the correction to the rr component:

$$g(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2} - \frac{q^2}{r^6}\left\{ \begin{array}{l} \quad\frac{4}{5}(d_2 + 4d_3)(6q^2 - 15mr + 10r^2) \\ +8d_4(3q^2 - 7mr + 4r^2) + \frac{4}{5}d_5(11q^2 - 25mr + 15r^2) \\ +\frac{4}{5}d_6(16q^2 - 35mr + 20r^2) + \frac{8}{5}(2d_7 + d_8)q^2 \end{array} \right\}$$

- f and g are required to have the same zeros, since otherwise there would be a non-Lorentzian spacetime region. Can confirm this via direct calculation.

Wald entropy formula

Wald entropy for black hole in IR EFT, for a spherically symmetric spacetime:

$$S = -2\pi A \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma}\Big|_{g_{\mu\nu}, r_{\rm H}}$$

Binormal to the horizon:

$$\epsilon_{\mu\nu} = \sqrt{\frac{f(r)}{g(r)}}(\delta^t_\mu \delta^r_\nu - \delta^r_\mu \delta^t_\nu)$$

Expand in perturbations:

- Horizon radius: $\rho = \tilde{\rho} + \Delta\rho$
- Area: $A = 4\pi\rho^2 = \widetilde{A} + \Delta A$
- Lagrangian: $\mathcal{L} = \widetilde{\mathcal{L}} + \Delta\mathcal{L}$

Wald entropy formula

Wald entropy for black hole in IR EFT, for a spherically symmetric spacetime:

$$S = -2\pi A \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} \bigg|_{g_{\mu\nu}, r_{\rm H}}$$

Expand the entropy:

$$S = -2\pi \left(\widetilde{A} \frac{\delta \widetilde{\mathcal{L}}}{\delta R_{\mu\nu\rho\sigma}} + \widetilde{A} \frac{\delta \Delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} + \Delta A \frac{\delta \widetilde{\mathcal{L}}}{\delta R_{\mu\nu\rho\sigma}} + \cdots \right) \epsilon_{\mu\nu} \epsilon_{\rho\sigma} \bigg|_{g_{\mu\nu}, \rho}$$

Wald entropy formula

Wald entropy for black hole in IR EFT, for a spherically symmetric spacetime:

$$S = -2\pi A \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma}\Big|_{g_{\mu\nu}, r_{\rm H}}$$

Expand the entropy:

$$S = -2\pi \left(\boxed{\widetilde{A}\,\frac{\delta \widetilde{\mathcal{L}}}{\delta R_{\mu\nu\rho\sigma}}} + \widetilde{A}\,\frac{\delta \Delta\mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} + \Delta A\,\frac{\delta \widetilde{\mathcal{L}}}{\delta R_{\mu\nu\rho\sigma}} + \cdots \right) \epsilon_{\mu\nu}\epsilon_{\rho\sigma}\Big|_{g_{\mu\nu},\,\rho}$$

$$\frac{\delta \widetilde{\mathcal{L}}}{\delta R_{\mu\nu\rho\sigma}} = \frac{1}{2\kappa^2} g^{\mu\rho} g^{\nu\sigma}$$ (symmetrization implied)

Since $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$, yields background $\widetilde{S} = \frac{2\pi}{\kappa^2}\widetilde{A} = \frac{\widetilde{A}}{4G}$

Wald entropy formula

Wald entropy for black hole in IR EFT, for a spherically symmetric spacetime:

$$S = -2\pi A \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma}\Big|_{g_{\mu\nu}, r_{\mathrm{H}}}$$

Expand the entropy:

$$S = -2\pi \left(\widetilde{A} \frac{\delta \widetilde{\mathcal{L}}}{\delta R_{\mu\nu\rho\sigma}} + \boxed{\widetilde{A} \frac{\delta \Delta\mathcal{L}}{\delta R_{\mu\nu\rho\sigma}}} + \Delta A \frac{\delta \widetilde{\mathcal{L}}}{\delta R_{\mu\nu\rho\sigma}} + \cdots \right) \epsilon_{\mu\nu}\epsilon_{\rho\sigma}\Big|_{g_{\mu\nu}, \rho}$$

"Interaction" contribution:

$$\boxed{\Delta S_{\mathrm{I}} = -2\pi \widetilde{A} \frac{\delta \Delta\mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma}\Big|_{\widetilde{g}_{\mu\nu}, \widetilde{\rho}}}$$

$$\Delta S = S - \widetilde{S} = \Delta S_{\mathrm{I}} + \Delta S_{\mathrm{H}}$$

Wald entropy formula

Wald entropy for black hole in IR EFT, for a spherically symmetric spacetime:

$$S = -2\pi A \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma}\Big|_{g_{\mu\nu}, r_{\rm H}}$$

Expand the entropy:

$$S = -2\pi \left(\widetilde{A} \frac{\delta \widetilde{\mathcal{L}}}{\delta R_{\mu\nu\rho\sigma}} + \widetilde{A} \frac{\delta \Delta\mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} + \boxed{\Delta A \frac{\delta \widetilde{\mathcal{L}}}{\delta R_{\mu\nu\rho\sigma}}} + \cdots \right) \epsilon_{\mu\nu}\epsilon_{\rho\sigma}\Big|_{g_{\mu\nu},\, \rho}$$

"Horizon" contribution:

$$\boxed{\Delta S_{\rm H} = -2\pi \Delta A \frac{\delta \widetilde{\mathcal{L}}}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma}\Big|_{\widetilde{g}_{\mu\nu},\, \widetilde{\rho}} = \frac{2\pi}{\kappa^2}\Delta A}$$

$$\Delta S = S - \widetilde{S} = \Delta S_{\rm I} + \Delta S_{\rm H}$$

Interaction contribution

Variation of the action with respect to the Riemann tensor:

$$\frac{\delta \Delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} = \quad 2c_1 R g^{\mu\rho} g^{\nu\sigma} + 2c_2 R^{\mu\rho} g^{\nu\sigma} + 2c_3 R^{\mu\nu\rho\sigma}$$
$$+ c_4 F_{\alpha\beta} F^{\alpha\beta} g^{\mu\rho} g^{\nu\sigma} + c_5 F^{\mu}{}_{\alpha} F^{\rho\alpha} g^{\nu\sigma} + c_6 F^{\mu\nu} F^{\rho\sigma}$$

(anti)symmetrization implied

Inputting our unperturbed background to compute ΔS_{I} to $\mathcal{O}(c_i)$, we have:

$$\Delta S_{\mathrm{I}} = \widetilde{S} \times \frac{2}{m^2(1+\xi)^3} \left[8d_3 - 2(1-\xi)(d_2 + 6d_3 + 2d_4 + d_5 + 2d_6)\right]$$

written in terms of the rescaled coefficients

Horizon contribution

Expand metric as $g(r) = \widetilde{g}(r) + \Delta g(r)$ and horizon radius $\rho = \tilde{\rho} + \Delta\rho$

Enforce horizon condition to compute horizon shift:

$$0 = g(\rho) = \widetilde{g}(\widetilde{\rho}) + \Delta g(\widetilde{\rho}) + \Delta\rho\,\partial_{\tilde{\rho}}\widetilde{g}(\widetilde{\rho}) \qquad \Longrightarrow \qquad \Delta\rho = -\frac{\Delta g(\widetilde{\rho})}{\partial_{\tilde{\rho}}\widetilde{g}(\widetilde{\rho})}$$

Shift in the horizon area:

$$\Delta A = A - \widetilde{A} = 8\pi\widetilde{\rho}\Delta\rho = -\frac{8\pi\widetilde{\rho}\Delta g(\widetilde{\rho})}{\partial_{\tilde{\rho}}\widetilde{g}(\widetilde{\rho})}$$

Inputting our unperturbed background to compute ΔS_{H} to $\mathcal{O}(c_i)$, we have:

$$\Delta S_{\mathrm{H}} = \widetilde{S} \times \frac{4(1-\xi)}{5m^2\xi(1+\xi)^3}\left[(1+4\xi)(d_2+4d_3+d_5+d_6)+10\xi d_4+2(1-\xi)(2d_7+d_8)\right]$$

Near-extremal limit

Note that $\Delta S_{\rm H}$ diverges in the strict $\xi \to 0$ limit

Physical origin: inner and outer horizons degenerate, so $\partial \widetilde{g}(\widetilde{\rho})/\partial \widetilde{\rho} = 0$

How small can we consistently take ξ ?

- Demanding $\Delta S \ll \widetilde{S} \quad \Longrightarrow \quad \xi \gg \dfrac{|d_i|}{m^2}$
- Can make this bound arbitrarily small by making BH arbitrarily large
- But recall that for wave function renormalization to be subdominant, we required:

$$\rho \ll \frac{1}{\kappa m_\phi^2}$$

- Since $d_i \sim \dfrac{1}{m_\phi^2}$, the bound on ξ becomes

$$\boxed{\xi \gg \kappa^2 m_\phi^2}$$

which can be made parametrically small for weakly coupled theories ($m_\phi \ll m_{\rm Pl}$)

Near-extremal limit

Note that $\Delta S_{\rm H}$ diverges in the strict $\xi \to 0$ limit

Physical origin: inner and outer horizons degenerate, so $\partial \widetilde{g}(\widetilde{\rho})/\partial \widetilde{\rho} = 0$

Further test: What about the temperature?

- We've checked that $\beta = \partial_M S$ for the perturbed black hole agrees with the surface gravity computed from the metric.
- For near-extremal black holes, $\widetilde{\beta} \sim m/\xi$ and $\Delta\beta \sim d_i/m\xi^3$
- Demanding $\Delta\beta \ll \widetilde{\beta} \Longrightarrow \xi \gg \dfrac{|d_i|^{1/2}}{m}$
- Again imposing $\rho \ll \dfrac{1}{\kappa m_\phi^2}$ implies

$$\boxed{\xi \gg \kappa m_\phi}$$

which we can still take parametrically small, since $m_\phi \ll m_{\rm Pl}$

New positivity bounds

General bounds

Total black hole entropy shift:

$$\Delta S = \widetilde{S} \times \frac{4}{5m^2\xi(1+\xi)^3} \times \\ \times \left[(1-\xi)^2(d_2+d_5) + 2(2+\xi+7\xi^2)d_3 + (1-\xi)(1-6\xi)d_6 + 2(1-\xi)^2(2d_7+d_8)\right]$$

Entropy bound $\Delta S > 0$ implies

$$(1-\xi)^2 d_0 + 20\xi d_3 - 5\xi(1-\xi)(2d_3+d_6) > 0$$

where

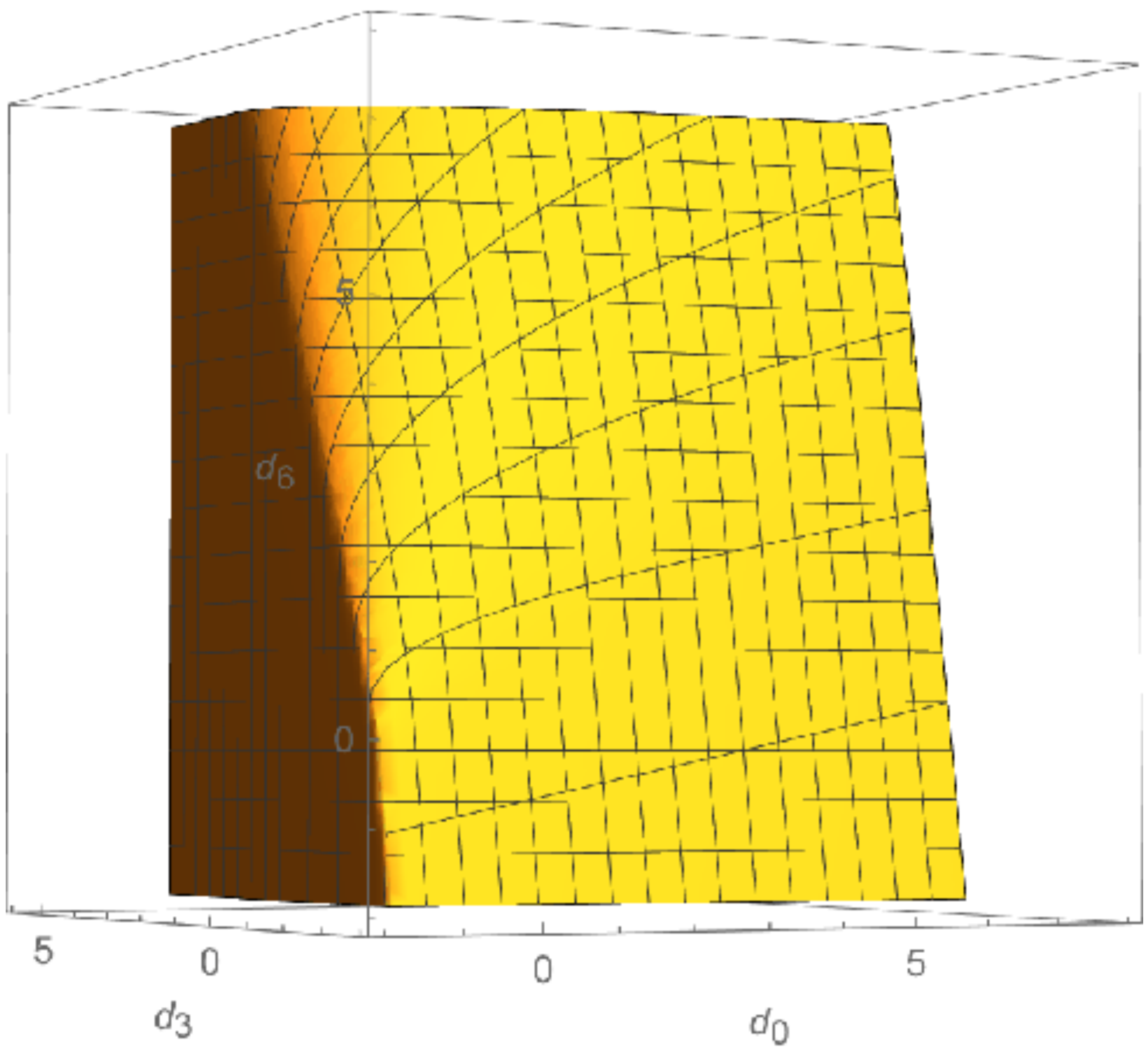
$$d_0 = d_2 + 4d_3 + d_5 + d_6 + 4d_7 + 2d_8$$

Coefficients are required to satisfy this bound for all values of $\xi \in (0, 1/2)$

Each value of ξ gives a linearly independent bound

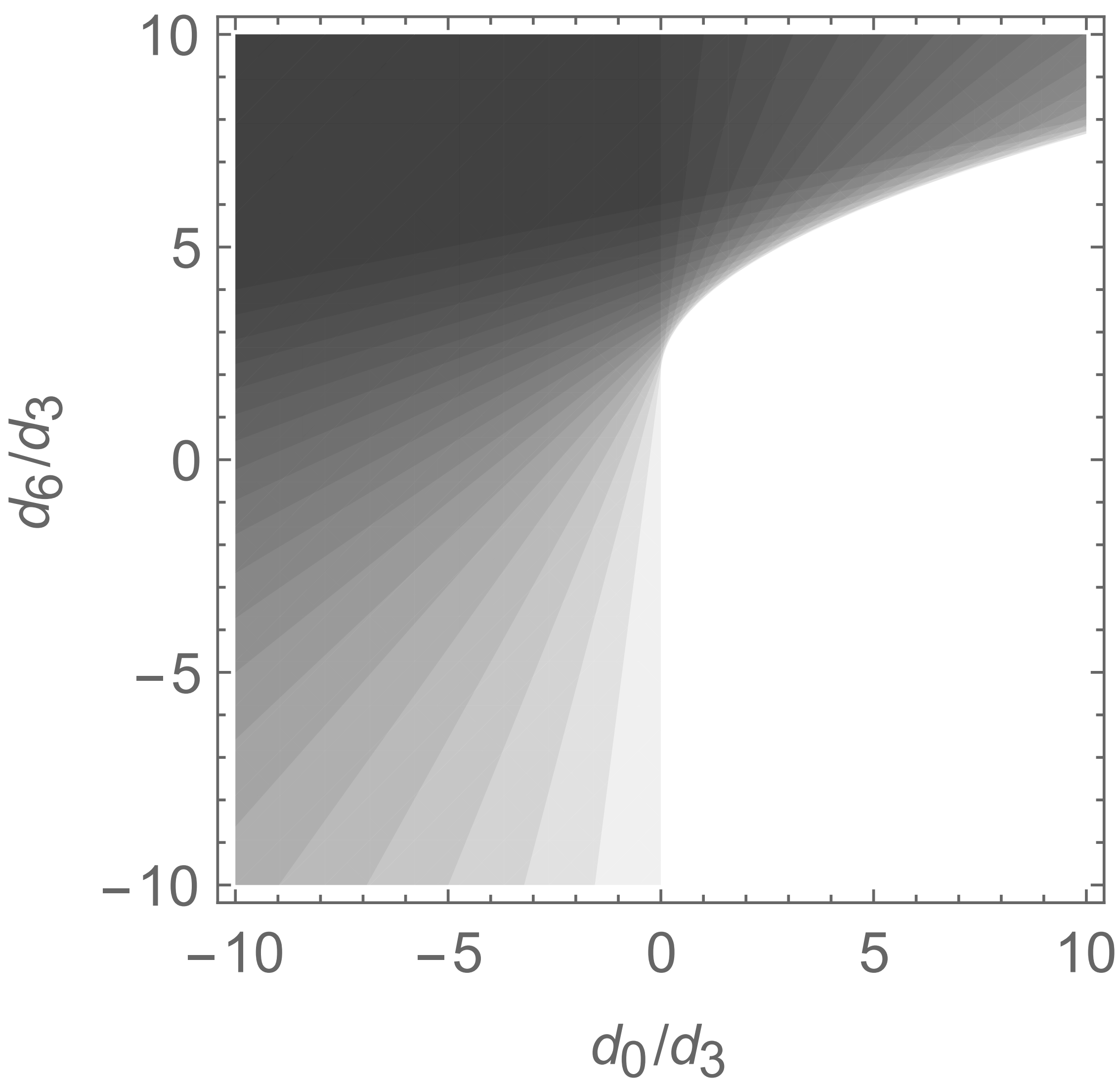
General bounds

Allowed region in d_0-d_3-d_6 space:

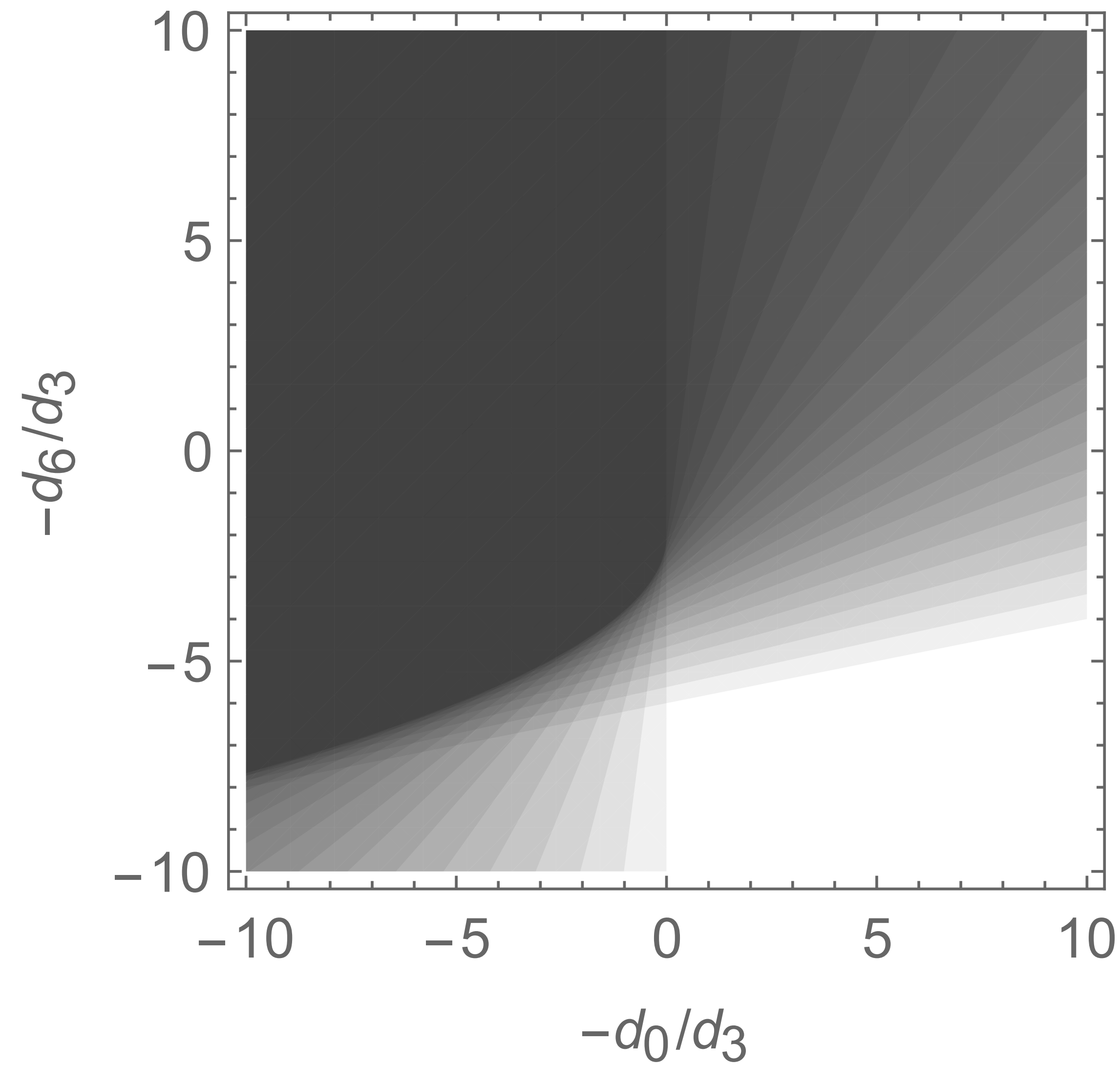


General bounds

Another visualization of the excluded regions:



$d_3 > 0$



$d_3 < 0$

The Weak Gravity Conjecture

In $\xi \ll 1$ (near-extremal) limit, the bound becomes

$$\boxed{d_0 > 0}$$

How is this connected to the Weak Gravity Conjecture?

- In Einstein-Maxwell theory + higher-curvature terms, the extra operators modify the allowed black hole charges
- Original, unperturbed extremality condition is $\widetilde{z} = \dfrac{q}{m} = 1$
- New extremality value is $z = 1 + \Delta z$
- Compute by imposing horizon condition:

$$0 = g(\rho, z) = \widetilde{g}(\widetilde{\rho}, \widetilde{z}) + \Delta g(\widetilde{\rho}, \widetilde{z}) + \Delta\rho\, \partial_{\widetilde{\rho}}\widetilde{g}(\widetilde{\rho}, \widetilde{z}) + \Delta z\, \partial_{\widetilde{z}}\widetilde{g}(\widetilde{\rho}, \widetilde{z})$$

$$\Delta z = -\frac{\Delta g(\widetilde{\rho}, \widetilde{z})}{\partial_{\widetilde{z}}\widetilde{g}(\widetilde{\rho}, \widetilde{z})}$$

The Weak Gravity Conjecture

In $\xi \ll 1$ (near-extremal) limit, the bound becomes

$$\boxed{d_0 > 0}$$

How is this connected to the Weak Gravity Conjecture?

- Direct computation:

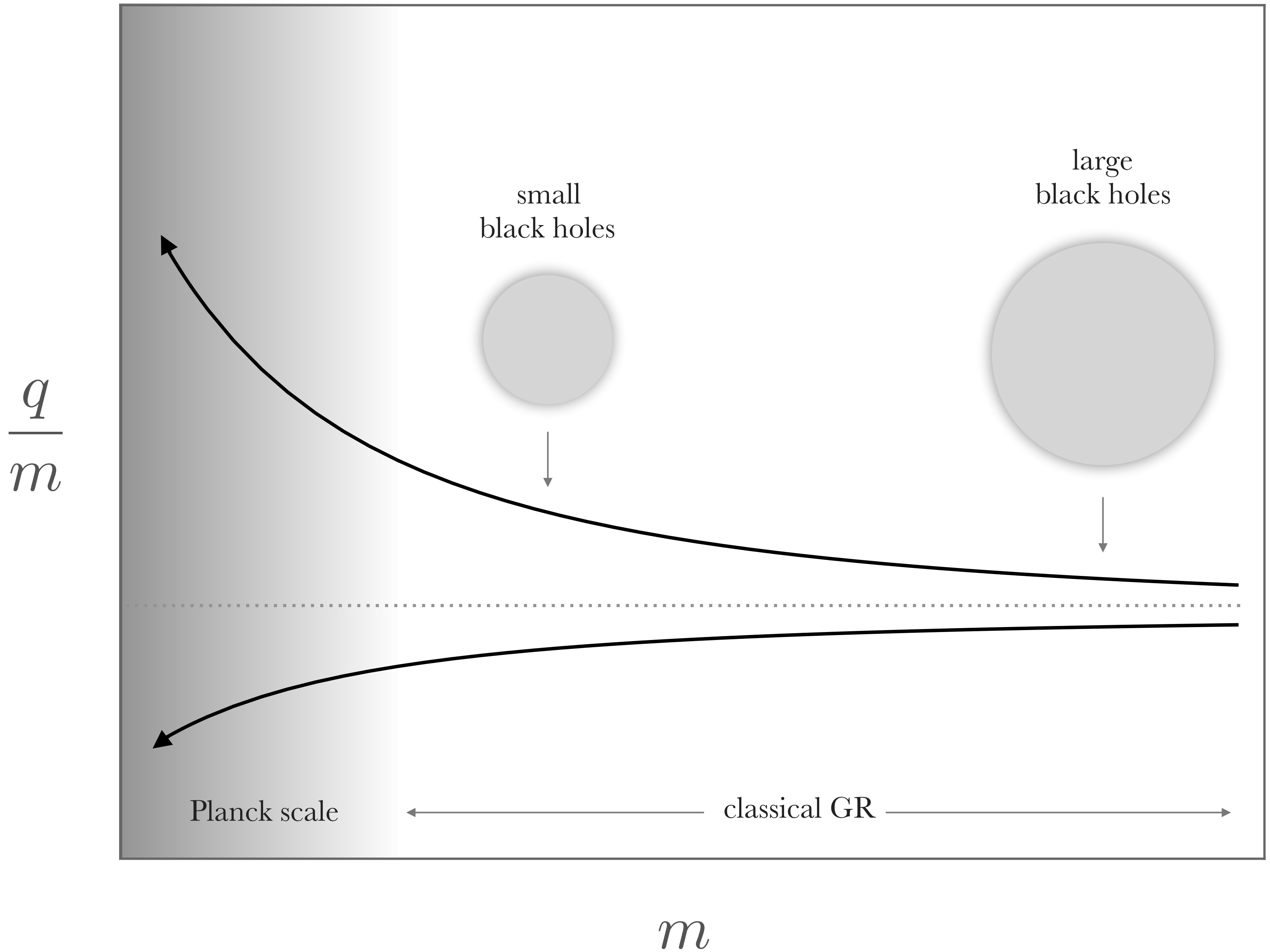
$$\boxed{\Delta z = \frac{2d_0}{5m^2} > 0}$$

Same combination of coefficients

Consistency of black hole entropy proves the Weak Gravity Conjecture.

- Since Δz grows as the BH gets smaller, extremal BHs can keep on decaying to yet lighter extremal black holes until they reach the scale of the UV completion.

The Weak Gravity Conjecture



- Since Δz grows as the BH gets smaller, extremal BHs can keep on decaying to yet lighter extremal black holes until they reach the scale of the UV completion.

Entropy, area, and extremity

Consistency of black hole entropy proves the Weak Gravity Conjecture.

Why did the same combination of coefficients d_0 appear?

$$0 = g(\rho, z) = \widetilde{g}(\widetilde{\rho}, \widetilde{z}) + \Delta g(\widetilde{\rho}, \widetilde{z}) + \Delta\rho\, \partial_{\widetilde{\rho}}\widetilde{g}(\widetilde{\rho}, \widetilde{z}) + \Delta z\, \partial_{\widetilde{z}}\widetilde{g}(\widetilde{\rho}, \widetilde{z})$$

- For near-extremal BH with q, m fixed, $\Delta z = 0$, so $\Delta\rho = -\Delta g / \partial_{\widetilde{\rho}}\widetilde{g}$
- For exactly extremal BH with free charge and mass, $\Delta z = -\Delta g / \partial_{\widetilde{z}}\widetilde{g}$
- Since radial component of metric is spacelike, $\partial_{\widetilde{\rho}}\widetilde{g} > 0$
- Metric dictates gravitational potential, which decreases with m, so $\partial_{\widetilde{z}}\widetilde{g} > 0$ so $\Delta\rho$ and Δz have the same sign
- Near-extremal entropy shift is dominated by horizon shift, so

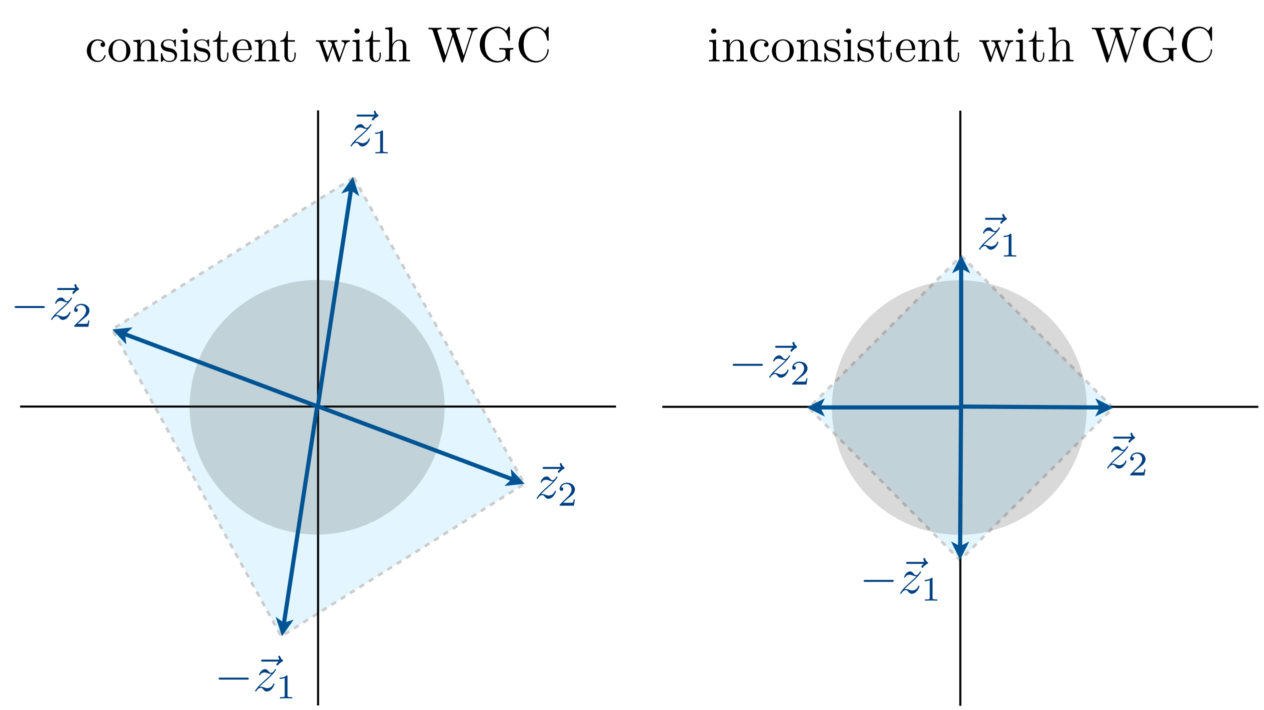
$$\Delta S \sim \Delta\rho \sim \Delta z > 0$$

Generalized Weak Gravity Conjecture

This logic generalizes to theories with multiple Abelian gauge fields:

Define vector $\mathbf{z}$ in charge-to-mass ratio space
All possible large BH states = unit ball

Generalized WGC: unit ball $\subset$ convex hull of lighter states   Cheung, GNR [1402.2287]



Generalized Weak Gravity Conjecture

This logic generalizes to theories with multiple Abelian gauge fields:

Define vector $\mathbf{z}$ in charge-to-mass ratio space
All possible large BH states = unit ball

Generalized WGC: unit ball $\subset$ convex hull of lighter states

Metric only depends on $\widetilde{z} = |\widetilde{\mathbf{z}}|$, so earlier argument applies,
using $\Delta z = \Delta \mathbf{z} \cdot \widetilde{\mathbf{z}} / |\widetilde{\mathbf{z}}|$, and implying

$$\Delta \rho > 0 \iff \Delta \mathbf{z} \cdot \widetilde{\mathbf{z}} > 0$$

Thus, for finite-mass, charged BH, the unit ball expands in all directions.

Generalized Weak Gravity Conjecture

This logic generalizes to theories with multiple Abelian gauge fields:

Define vector $\mathbf{z}$ in charge-to-mass ratio space
All possible large BH states = unit ball

Generalized WGC: unit ball $\subset$ convex hull of lighter states

Metric only depends on $\widetilde{z} = |\widetilde{\mathbf{z}}|$, so earlier argument applies,
using $\Delta z = \Delta \mathbf{z} \cdot \widetilde{\mathbf{z}}/|\widetilde{\mathbf{z}}|$, and implying

$$\Delta\rho > 0 \iff \Delta\mathbf{z} \cdot \widetilde{\mathbf{z}} > 0$$

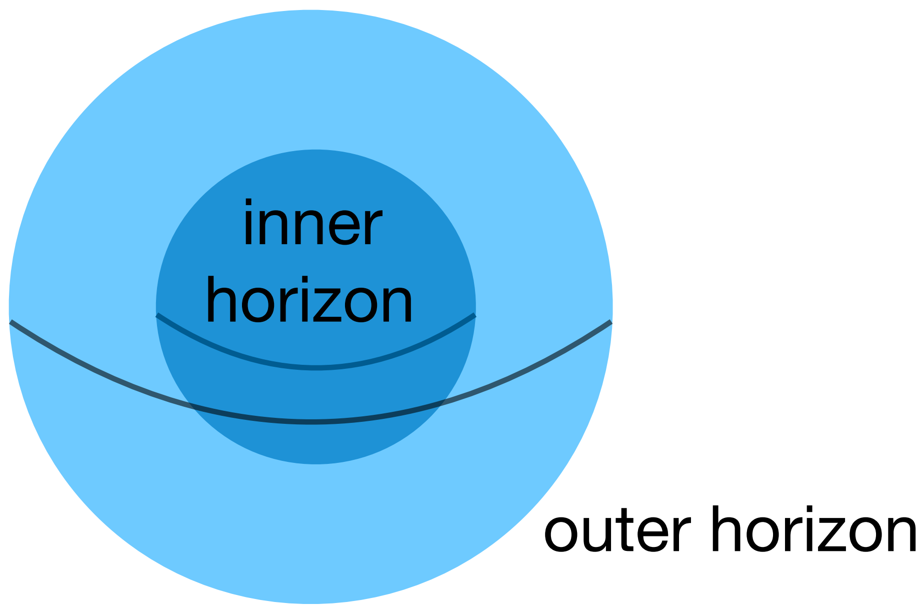
Thus, for finite-mass, charged BH, the unit ball expands in all directions.

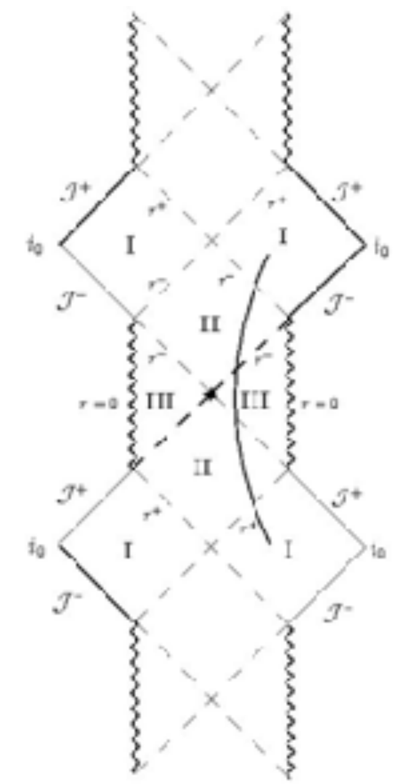
Consistency of black hole entropy proves the generalized Weak Gravity Conjecture.

Generalization to arbitrary dimension

Unperturbed solution

The Reissner-Nordström black hole:





Static, spherically-symmetric metric:

$$\mathrm{d}s^2 = \widetilde{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\widetilde{f}(r)\mathrm{d}t^2 + \frac{1}{\widetilde{g}(r)}\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{D-2}$$

Unperturbed components:

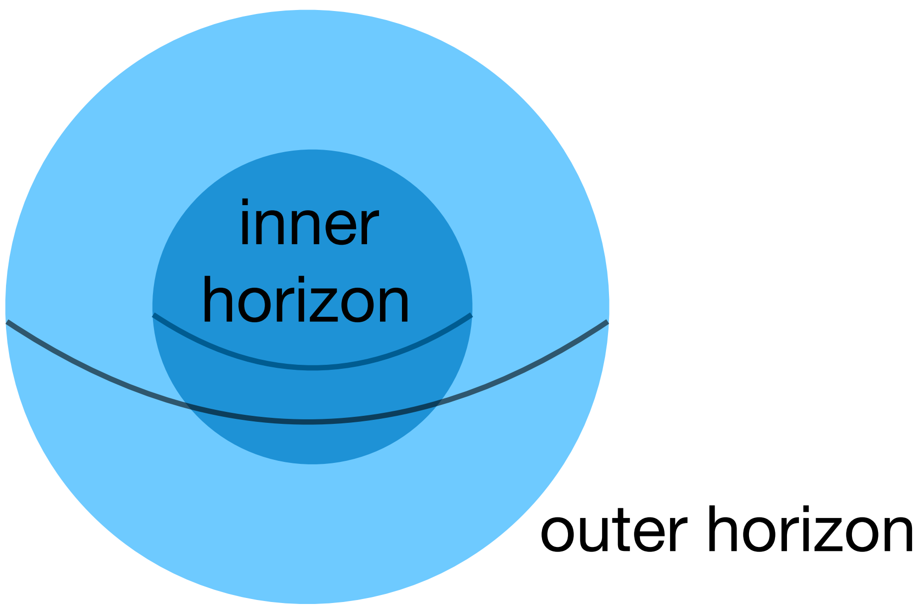
$$\widetilde{f}(r) = \widetilde{g}(r) = 1 - \frac{2\kappa^2 M}{(D-2)\Omega_{D-2} r^{D-3}} + \frac{Q^2\kappa^2}{(D-2)(D-3)\Omega^2_{D-2} r^{2(D-3)}}$$

Field strength:

$$\widetilde{F} = \frac{Q}{\Omega_{D-2} r^{D-2}} \mathrm{d}t \wedge \mathrm{d}r$$

Unperturbed solution

The Reissner-Nordström black hole:



Convenient units:

$$m = \frac{\kappa^2 M}{(D-2)\Omega_{D-2}}$$

$$q = \frac{\kappa Q}{\sqrt{(D-2)(D-3)}\Omega_{D-2}}$$

Redefined radial coordinate:

$$x = r^{D-3}$$

Unperturbed components:

$$\widetilde{f}(r) = \widetilde{g}(r) = 1 - \frac{2m}{x} + \frac{q^2}{x^2}$$

Field strength:

$$\widetilde{F} = \frac{Q}{\Omega_{D-2} r^{D-2}} \mathrm{d}t \wedge \mathrm{d}r$$

Outer (event) horizon:

$$x = \widetilde{\chi}$$

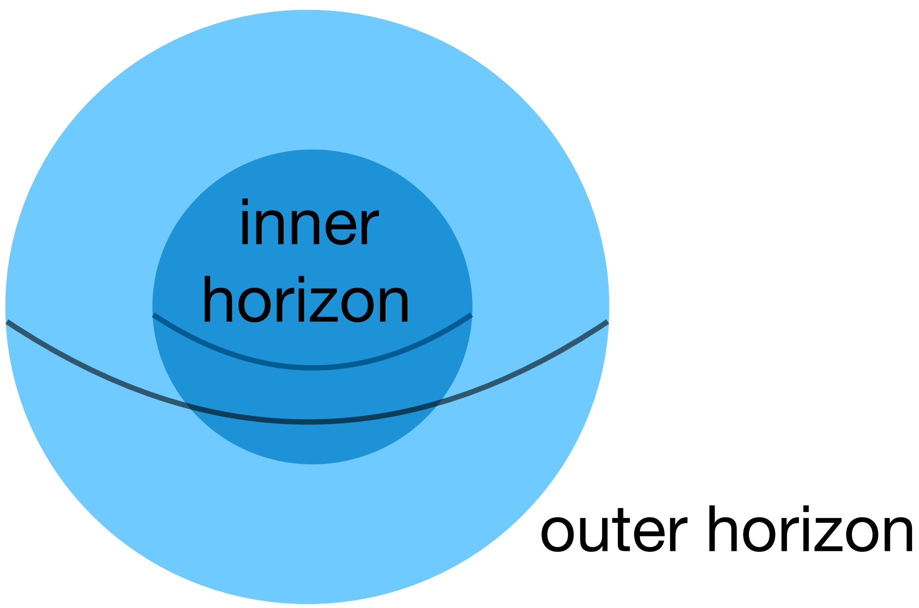
$$\widetilde{\chi} = m + \sqrt{m^2 - q^2}$$
$$= m(1+\xi)$$

Extremality condition:

$$\frac{q}{m} \leq 1$$

Unperturbed solution

The Reissner-Nordström black hole:



Convenient units:

$$m = \frac{\kappa^2 M}{(D-2)\Omega_{D-2}}$$

$$q = \frac{\kappa Q}{\sqrt{(D-2)(D-3)}\Omega_{D-2}}$$

Redefined radial coordinate:

$$x = r^{D-3}$$

Unperturbed components:

$$\widetilde{f}(r) = \widetilde{g}(r) = 1 - \frac{2m}{x} + \frac{q^2}{x^2}$$

Field strength:

$$\widetilde{F} = \frac{Q}{\Omega_{D-2} r^{D-2}} \mathrm{d}t \wedge \mathrm{d}r$$

Thermodynamic stability (positive specific heat) requires:

$$\frac{q}{m} > \frac{\sqrt{2D-5}}{D-2}$$

$$\Downarrow$$

$$\xi < \frac{D-3}{D-2}$$

Perturbed charged black hole metric

Inversion of the Ricci tensor works the same as before:

$$g(r) = 1 - \frac{2\kappa^2 M}{(D-2)\Omega_{D-2} r^{D-3}} - \frac{2\kappa^2}{(D-2) r^{D-3}} \int_r^{+\infty} \mathrm{d}r \; r^{D-2} {T^t}_t$$

$$f(r) = g(r) \exp\left[\frac{2\kappa^2}{D-2} \int_r^{+\infty} \mathrm{d}r \frac{r}{g(r)} ({T^t}_t - {T^r}_r)\right]$$

Again, compute corrections to metric by treating higher-order terms as perturbations $\Delta T_{\mu\nu}$

We find:

$$g(r) = 1 - \frac{2m}{x} + \frac{q^2}{x^2} - \frac{q^2}{x^{\frac{2(2D-5)}{D-3}}} \sum_{i=1}^{8} \alpha_i(x) c_i$$

Perturbed charged black hole metric

$$g(r) = 1 - \frac{2m}{x} + \frac{q^2}{x^2} - \frac{q^2}{x^{\frac{2(2D-5)}{D-3}}} \sum_{i=1}^{8} \alpha_i(x) c_i$$

where

$$\alpha_1 = \frac{(D-3)(D-4)}{D-2} \left[2\frac{13D^2 - 47D + 40}{3D-7} q^2 - 8(3D-5)mx + 16(D-2)x^2 \right]$$

$$\alpha_2 = 2\frac{D-3}{D-2} \left[\frac{8D^3 - 55D^2 + 117D - 76}{3D-7} q^2 - 4(2D^2 - 10D + 11)mx \right.$$
$$\left. + 2(3D-10)(D-2)x^2 \right]$$

$$\alpha_3 = 4\frac{D-3}{D-2} \left[\frac{8D^3 - 48D^2 + 87D - 44}{3D-7} q^2 - 2(4D^2 - 17D + 16)mx \right.$$
$$\left. + 8(D-2)(D-3)x^2 - 2(D-2)(D-4)\frac{m^2x^2}{q^2} \right]$$

$$\alpha_4 = 4(D-3) \left[\frac{(7D-13)(D-2)}{3D-7} q^2 - 2(3D-5)mx + 4(D-2)x^2 \right]$$

Perturbed charged black hole metric

$$g(r) = 1 - \frac{2m}{x} + \frac{q^2}{x^2} - \frac{q^2}{x^{\frac{2(2D-5)}{D-3}}} \sum_{i=1}^{8} \alpha_i(x) c_i$$

where

$$\alpha_5 = 2(D-3)\left[\frac{(5D-9)(D-2)}{3D-7} q^2 - 2(2D-3)mx + 3(D-2)x^2\right]$$

$$\alpha_6 = 4(D-3)\left[4\frac{(D-2)^2}{3D-7} q^2 - (3D-5)mx + 2(D-2)x^2\right]$$

$$\alpha_7 = 8\frac{(D-2)(D-3)^2}{3D-7} q^2$$

$$\alpha_8 = 4\frac{(D-2)(D-3)^2}{3D-7} q^2$$

Calculation of entropy

As in $D = 4$, split entropy shift into contributions from interactions in Wald formula and from the shift in the horizon location:

$$\Delta S = S - \widetilde{S} = \Delta S_{\mathrm{I}} + \Delta S_{\mathrm{H}}$$

$$\Delta S_{\mathrm{I}} = -2\pi \widetilde{A}\, \frac{\delta \Delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma}\Bigg|_{\widetilde{g}_{\mu\nu},\, \widetilde{\rho}}$$

Direct calculation yields:

$$\begin{aligned}\Delta S_{\mathrm{I}} = \widetilde{S} \times \frac{2(D-3)}{m^{\frac{2}{D-3}}(1+\xi)^{\frac{D-1}{D-3}}} \Big\{ & 4(D-2)d_3 \\ & - 2(1-\xi)\left[(D-4)d_1 + (D-3)d_2 + 2(2D-5)d_3 + (D-2)\left(d_4 + \frac{1}{2}d_5 + d_6\right)\right]\Big\}\end{aligned}$$

Calculation of entropy

As in $D = 4$, split entropy shift into contributions from interactions in Wald formula and from the shift in the horizon location:

$$\Delta S = S - \widetilde{S} = \Delta S_{\mathrm{I}} + \Delta S_{\mathrm{H}}$$

$$\Delta S_{\mathrm{H}} = -2\pi \Delta A \, \frac{\delta \widetilde{\mathcal{L}}}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} \Bigg|_{\widetilde{g}_{\mu\nu},\, \widetilde{\rho}} = \frac{2\pi}{\kappa^2} \Delta A$$

where

$$\Delta A = A - \widetilde{A} = (D-2)\Omega_{D-2} \widetilde{\rho}^{D-3} \Delta\rho = -\frac{(D-2)\Omega_{D-2} \widetilde{\chi} \Delta g(\widetilde{\rho})}{\partial \widetilde{g}(\widetilde{\rho}) / \partial \widetilde{\rho}}$$

Calculation of entropy

As in $D = 4$, split entropy shift into contributions from interactions in Wald formula and from the shift in the horizon location:

$$\Delta S = S - \widetilde{S} = \Delta S_{\mathrm{I}} + \Delta S_{\mathrm{H}}$$

Direct calculation yields:

$$\begin{aligned}
\Delta S_{\mathrm{H}} = \widetilde{S} \times \frac{1}{(3D-7)m^{\frac{2}{D-3}}\xi(1+\xi)^{\frac{D-1}{D-3}}} \times \\
\times \Big\{ & d_1(1-\xi)(D-3)(D-4)[(11D-24)\xi + D - 4] \\
& + d_2(1-\xi)(D-3)[(10D^2-53D+68)\xi + 2D^2 - 11D + 16] \\
& + 2d_3[-(16D^3 - 128D^2 + 337D - 292)(1-\xi)^2 \\
& \qquad + 2(3D-7)(4D^2-23D+32)(1-\xi) \\
& \qquad - 2(D-2)(D-4)(3D-7)] \\
& + 2d_4(1-\xi)(D-2)(D-3)[5(D-2)\xi + D - 4] \\
& + 2(d_5+d_6)(D-2)(D-3)(1-\xi)[2(D-2)\xi + D - 3] \\
& + 2(2d_7+d_8)(D-2)^2(D-3)(1-\xi)^2 \Big\}
\end{aligned}$$

Near-extremal limit

Note that $\Delta S_{\rm H}$ diverges in the strict $\xi \to 0$ limit

Physical origin: inner and outer horizons degenerate, so $\partial\widetilde{g}(\widetilde{\rho})/\partial\widetilde{\rho} = 0$

How small can we consistently take ξ ?

- Demanding $\Delta S \ll \widetilde{S} \quad \Longrightarrow \quad \xi \gg \dfrac{|d_i|}{m^{\frac{2}{D-3}}}$
- Can make this bound arbitrarily small by making BH arbitrarily large
- But recall that for wave function renormalization to be subdominant, we required:

$$\rho \ll \frac{1}{\kappa m_\phi^{D/2}}$$

- Since $d_i \sim \dfrac{1}{m_\phi^2}$, the bound on ξ becomes

$$\boxed{\xi \gg \kappa^2 m_\phi^{D-2}}$$

which can be made parametrically small for weakly coupled theories ($m_\phi \ll m_{\rm Pl}$)

Near-extremal limit

Note that $\Delta S_{\rm H}$ diverges in the strict $\xi \to 0$ limit

Physical origin: inner and outer horizons degenerate, so $\partial\widetilde{g}(\widetilde{\rho})/\partial\widetilde{\rho} = 0$

Further test: What about the temperature?

- We've checked that $\beta = \partial_M S$ for the perturbed black hole agrees with the surface gravity computed from the metric.
- Background inverse temperature: $\widetilde{\beta} = \dfrac{2\pi m^{\frac{1}{D-3}}(1+\xi)^{\frac{D-2}{D-3}}}{(D-3)\xi}$
- Inverse temperature shift for near-extremal BH: $\Delta\beta \sim d_i/\xi^3 m^{1/(D-3)}$
- Demanding $\Delta\beta \ll \widetilde{\beta} \implies \xi \gg \dfrac{|d_i|^{1/2}}{m^{\frac{1}{D-3}}}$
- Again imposing $\rho \ll \dfrac{1}{\kappa m_\phi^{D/2}}$ implies

$$\boxed{\xi \gg \kappa m_\phi^{(D-2)/2}}$$

which we can still take parametrically small, since $m_\phi \ll m_{\rm Pl}$

New positivity bounds

Total black hole entropy shift:

$$
\begin{aligned}
\Delta S = \widetilde{S} \times \frac{1}{(3D-7)m^{\frac{2}{D-3}}\xi(1+\xi)^{\frac{D-1}{D-3}}} \times \\
\times \Big\{ & d_1(D-3)(D-4)^2(1-\xi)^2 \\
& + d_2(D-3)(2D^2-11D+16)(1-\xi)^2 \\
& + 2d_3\big[(8D^3-60D^2+151D-128)(1-\xi)^2 \\
& \qquad -2(D-2)(2D-5)(3D-7)(1-\xi) \\
& \qquad +2(D-2)^2(3D-7)\big] \\
& + 2d_4(D-2)(D-3)(D-4)(1-\xi)^2 \\
& + 2d_5(D-2)(D-3)^2(1-\xi)^2 \\
& + 2d_6(D-2)(D-3)(1-\xi)[-2(2D-5)\xi+D-3] \\
& + 4d_7(D-2)^2(D-3)(1-\xi)^2 \\
& + 2d_8(D-2)^2(D-3)(1-\xi)^2 \Big\}
\end{aligned}
$$

General bounds

Entropy bound $\Delta S > 0$ implies

$$(1-\xi)^2 d_0 + (D-2)^2(3D-7)\xi d_3 - \frac{1}{2}(D-2)(D-3)(3D-7)\xi(1-\xi)(2d_3+d_6) > 0$$

where

$$\begin{aligned} d_0 = &\frac{1}{4}(D-3)(D-4)^2 d_1 + \frac{1}{4}(D-3)(2D^2-11D+16)d_2 \\ &+ \frac{1}{2}(2D^3-16D^2+45D-44)d_3 + \frac{1}{2}(D-2)(D-3)(D-4)d_4 \\ &+ \frac{1}{2}(D-2)(D-3)^2(d_5+d_6) + (D-2)^2(D-3)\left(d_7+\frac{1}{2}d_8\right) \end{aligned}$$

Coefficients are required to satisfy this bound for all values of $\xi \in \left(0, \frac{D-3}{D-2}\right)$

Each value of ξ gives a linearly independent bound

New positivity bounds

As before, taking the near-extremal ($\xi \ll 1$) limit implies

$$\boxed{d_0 > 0}$$

The shift in extremality condition of the black hole in D dimensions is

$$\boxed{\Delta z = \frac{4(D-3)}{(3D-7)(D-2)m^{\frac{2}{D-3}}} d_0}$$

same combination of coefficients

Again, we find:

Consistency of black hole entropy proves the Weak Gravity Conjecture.

Examples and consistency checks

Field redefinition invariance

Any physical observable should be invariant under a reparameterization of the field variables, e.g.,

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} = g_{\mu\nu} + r_1 R_{\mu\nu} + r_2 g_{\mu\nu} R + r_3 \kappa^2 F_{\mu\rho} F_{\nu}{}^{\rho} + r_4 \kappa^2 g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$$

This has the effect of shifting the action, $\delta \mathcal{L} = \frac{1}{2\kappa^2} \delta g^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \kappa^2 T_{\mu\nu} \right)$

which has the net effect of shifting the higher-dimension operator coefficients:

$$d_1 \to d_1 - \frac{1}{4} r_1 - \frac{D-2}{4} r_2$$

$$d_2 \to d_2 + \frac{1}{2} r_1$$

$$d_3 \to d_3$$

$$d_4 \to d_4 + \frac{1}{8} r_1 + \frac{D-4}{8} r_2 - \frac{1}{4} r_3 - \frac{D-2}{4} r_4$$

$$d_5 \to d_5 - \frac{1}{2} r_1 + \frac{1}{2} r_3$$

$$d_6 \to d_6$$

$$d_7 \to d_7 + \frac{1}{8} r_3 + \frac{D-4}{8} r_4$$

$$d_8 \to d_8 - \frac{1}{2} r_3$$

Field redefinition invariance

There are four combinations of higher-dimension operator coefficients that are invariant under this transformation:

$$
\begin{aligned}
d_0 = {} & \frac{1}{4}(D-3)(D-4)^2 d_1 + \frac{1}{4}(D-3)(2D^2-11D+16)d_2 \\
& + \frac{1}{2}(2D^3-16D^2+45D-44)d_3 + \frac{1}{2}(D-2)(D-3)(D-4)d_4 \\
& + \frac{1}{2}(D-2)(D-3)^2(d_5+d_6) + (D-2)^2(D-3)\left(d_7 + \frac{1}{2}d_8\right)
\end{aligned}
$$

$$d_3$$

$$d_6$$

$$d_9 = d_2 + d_5 + d_8$$

The total entropy shift ΔS, and hence our bounds, are built out of d_0, d_3, d_6, and hence are field redefinition invariant.

Concrete examples

1. Only photon self-interactions ($d_{7,8}$). Our bound becomes simply $2d_7 + d_8 > 0$. When we compute the four-photon scattering amplitude and apply the analyticity arguments of Adams et al. [hep-th/0602178], we find that different choices of photon polarizations give $2d_7 + d_8 > 0$ and $d_8 > 0$, so this is consistent.

2. Scalar completion:
$$\mathcal{L} = \frac{1}{2\kappa^2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left(\frac{a_\phi}{\kappa}R + b_\phi \kappa F_{\mu\nu}F^{\mu\nu}\right)\phi - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2}m_\phi^2\phi^2$$
generates
$$d_i = \frac{1}{2m_\phi^2} \times \{a_\phi^2, 0, 0, 2a_\phi b_\phi, 0, 0, b_\phi^2, 0\}$$
so
$$d_0 = \frac{D-3}{8m_\phi^2}\left[(D-4)a_\phi + 2(D-2)b_\phi\right]^2 > 0$$

3. Low-energy description of the heterotic string: Kats, Motl, Padi [hep-th/0606100]; Gross, Sloan (1987)
$$d_i = \frac{\alpha'}{64} \times \{4, -16, 4, 0, 0, 0, -3, 12\}$$
Our bound then becomes $(6D^2 - 30D + 37)\xi^2 + 2(D-2)\xi + 2D - 5 > 0$, which is satisfied for all $\xi \in (0,1)$ and $D > 3$.

Discussion and conclusions

Discussion

- In this work, we relied on a universal notion of thermodynamic entropy: $\Delta S > 0$ when more microstates are added to a system of a given macrostate, which we proved for tree-level completions in QFT

- Applying this logic to the system of charged black holes, we can compare the Wald and Bekenstein-Hawking entropy in the Einstein-Maxwell EFT

- Imposing the entropy bound requires positivity of various combinations of higher-dimension operator couplings R^2, RF^2, and F^4, producing a family of bounds labeled by ξ

- For a near-extremal BH, these bounds imply positivity of the same combination of coefficients that also guarantees a positive correction to the extremality bound for BHs in the EFT

- Thus, consistency of BH entropy proves the WGC

- Generalizes to multiple gauge fields and arbitrary dimension

Future directions

- Can other swampland program bounds be derived using black hole entropy?
  - Broader class of theories, e.g., Einstein-dilaton gravity
  - Other metrics: (A)dS-black hole, non-spherical metrics, etc.
- More broadly, understand the relationship between entropy bounds and bounds from analyticity, unitarity, and causality
  - Positivity of entropy shifts comes from UV state-counting, reminiscent of bounds from dispersion relations and spectral representations
- Extended versions of the WGC?
- Much work remains in separating the swampland from the landscape and new tools continue to be discovered